# Numerical Analysis Qualifying Written Exam (January 2023)

# Part I: do 3 of 4

**1.** Consider a floating point system $F(\beta, t, m, M)$, where $m$, $M$ are the minimum and the maximum exponent respectively.
**(a)** Prove that there are as many numbers between 0 and 1 as between 1 and $\infty$.
**(b)** Define when an algorithm is backward stable.
**(c)** Show that the addition of two floating point numbers is a backward stable operation.
**(c)** Show that the division of two floating point numbers is a backward stable operation.

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**2.**
**(a)** Explain the secant method and Newton's method for solving root finding problems.
**(b)** How are the two methods connected? Explain.
**(c)** What can you say about the convergence properties of the two methods? You can use illustrations and/or proofs to explain your conclusions.

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**3.**
**(a)** Explain the difference between Boundary Value Problems (BVP) and Initial Value Problems (IVP).
**(b)** Given the BVP

$$\begin{cases} u'' + u = 0, & \text{in } ]0, \pi[ \\ u(0) = 0,\ u(\pi) = 1, & \end{cases}$$

find the exact solution $u$ and derive a numerical method to compute the approximate solution $u_h$ for the BVP.
**(c)** Explain how you can control numerical accuracy using above method and the order of accuracy of your method.

**4.**
**(a)** Using Taylor expansion, derive the leapfrog method

$$U^{n+1} = U^{n-1} + 2kf(U^n),$$

for solving Initial Value Problems (IVP), where $U^n$ is the numerical approximation of $u$ at time step $n$ and $k$ is the time step size.
**(b)** Calculate the local truncation error of the leapfrog method.
**(c)** How does this method differ from the Forward Euler method?

# Part II: do 2 of 3

**1.**
**(a)** Consider the standard trapezoidal rule for numerical integration $T \approx \int_a^b f(x)dx$. (It can be obtained by using an interpolating polynomial using $f(a)$ and $f(b)$). Give a formula for $T$ and one for the error $T - \int_a^b f(x)dx$. Give also an expression for the the composite rule ($N$ points in $(a, b)$), and the corresponding error.
**(b)** Consider now an Hermite interpolating polynomial $p(x)$ such that $p(a) = f(a)$, $p(b) = f(b)$, $p'(a) = f'(a)$, $p'(b) = f'(b)$. Construct a quadrature formula $H = \int_a^b p(x)dx \approx \int_a^b f(x)dx$. Give the expression for $H$ and compute the corresponding error $H - \int_a^b f(x)dx$. Give also an expression for the the composite rule ($N$ points in $(a, b)$), and the corresponding error.

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**2.**
**(a)** Show that the Crank-Nicholson (CN) method

$$U^{n+1} = U^n + k\left[\frac{1}{2}f(U^n) + \frac{1}{2}f(U^n + 1)\right]$$

is zero-stable.
**(b)** What is the definition of the *region of absolute stability* (RAS). Derive and draw the RAS for CN.
**(c)** Compare the RAS of CN and Forward Euler and explain the computational consequences stemming from the differences between the two RAS.

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**3.**
**(a)** Explain the difference between stiff and non-stiff problems.
**(b)** Would you use the Crank-Nicholson or Backward Euler method to solve a stiff problem? Explain using your knowledge of A- and L-stability and absolute stability.