Numerical Analysis Qualifying Written Exam (August 2023)

Part I: do 3 of 4

1. Consider a floating point system $F(\beta, t, m, M)$, where m, M are the minimum and the maximum exponent, respectively.
(a) What is the value of $\mathbf{u}$, the unit of roundoff, in terms of the above quantities?
(b) Consider depositing the amount c every day with an interest rate i compounded daily. With the accumulated capital, the total at the end of the year equals

$$c[(1+x)^n - 1]/x, \quad x = i/n \ll 1,$$

where $n = 365$.
Using your knowledge of floating point arithmetic, how would you compute this quantity so as to avoid excessive roundoff?

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2. Adaptive integration.
(a) Give the general principle of adaptive integration. Choose a particular method, e.g., Simpson's rule, and indicate how to implement the adaptive integration. Give formulas.
(b) Give examples of functions for which adaptive integration would be beneficial and of other functions for which regular quadrature would be faster. Explain your answers.

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3. Given the ODE Initial Value Problem $u'(t) = f(u(t))$, $u(0) = u_0$, $t \in [0, T]$.
(a) Write the Forward Euler (FE) method and the Backward Euler (BE) method.
(b) For the BE method, write how one time step (i.e., given U^n at t_n, find U^{n+1} at $t_n + k$) would look like if the resulting implicit relationship is approximated via 4 Newton iteration steps per time step (you may write a pseudo-code with a for-loop).
(c) Define, calculate, and draw the regions of absolute stability for FE and for BE.
(d) Calculate the local truncation errors of FE and of BE.
(e) Based on the aspects above, provide one important advantage of FE over BE.
(f) Based on the aspects above, provide one important advantage of BE over FE.

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4. Consider the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1/2 & 1/4 & 1/4 & \\ 1 & 1/3 & 1/3 & 1/3 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

(a) Write out one step of the method (i.e., given U^n at t_n, find U^{n+1} at $t_n + k$) that is defined by the above tableau.
(b) Prove that this method is at least second-order accurate.
(c) Provide the definitions of A-stability and of L-stability.
(d) Prove that this method is L-stable.
(e) How does the computational cost (per step) of this method compare to the cost of the backward Euler method?

Part II: do 2 of 3

1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ and consider the problem of finding a root of this function, i.e., a vector x_* such that $F(x_*) = 0$.
(a) Describe Newton's method for the solution of the problem, give formulas, starting with an initial vector x_0.
(b) Give appropriate conditions on F, x_0 and x_* to guarantee quadratic convergence. Define linear and quadratic convergence, and prove a theorem with your hypotheses, showing this convergence.
(c) Indicate what happens if each of the conditions in (b) do not hold. Indicate in which cases there is no convergence, and in which cases the convergence is linear.
(d) Give an example of a function F for which convergence is quadratic, and an example of a function for which the convergence is linear.

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2. Stiff problems.
(a) Explain what a stiff problem is. In which sense is it "harder" to numerically approximate than a non-stiff problem?
(b) Provide an example of a stiff problem.
(c) Write the Crank-Nicolson method, first for a general ODE IVP, then applied to your specific example from **(b)**.
(d) State whether the Crank-Nicolson method is A-stable and whether it is L-stable (you do not need to prove these properties; it suffices to simply state the fact and/or provide an intuitive argument).
(e) Based on the properties in **(d)**, explain: for what types of stiff problems is the Crank-Nicolson scheme a good choice, and under which circumstances could it not be a satisfactory method?

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3. Consider the following time-stepping scheme:
$U^{n+2} - \frac{4}{3}U^{n+1} + \frac{1}{3}U^n = \frac{2}{3}f(U^{n+2}, t_{n+2})$
(a) Define zero-stability and prove that the method is zero-stable.
(b) Prove that the scheme is A-stable. [Hint: You could (but do not have to) use the boundary locus method. In that case, you may use the fact that the curve $\frac{3}{2} - 2e^{i\theta} + \frac{1}{2}e^{2i\theta}$ does not penetrate the left half plane.]
(c) For the test problem $u' = \lambda u$ with $\lambda \in \mathbb{R}^-$, calculate the behavior of the scheme's (worst case) growth mode in the limit $k \to \infty$. Explain in which sense the scheme's behavior is superior to the growth factor of the Crank-Nicolson scheme in the same situation.