# Comprehensive Examination in Geometry & Topology
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### Part I. Solve three of the following problems.

**I.1** Let $V$ be a solid torus with meridian and longitude curves $\mu_V, \lambda_V$. (Recall that $\mu_V$ and $\lambda_V$ are characterized by the following properties: $\mu_V$ bounds a disk in $V$ and $\lambda_V$ generates $H_1(V)$.) Let $W$ be a solid torus with meridian and longitude curves $\mu_W, \lambda_W$. Let $M$ be a 3–manifold obtained by gluing $V$ to $W$ via a homeomorphism $\varphi : \partial V \to \partial W$ such that

$$\varphi_*(\mu_V) = 3\mu_W + 4\lambda_W, \qquad \varphi_*(\lambda_V) = 2\mu_W + 3\lambda_W.$$

Use the Mayer–Vietoris sequence to compute the homology groups $H_i(M)$.

**I.2** Let $M$ be an $(n-1)$-dimensional, compact submanifold of $\mathbb{R}^n$ not containing 0. Show that for all but a measure zero set of unit vectors $v \in S^{n-1}$, the ray $\{tv : t \geq 0\}$ meets $M$ in (at most) finitely many points.

**I.3** Let $M$ be a smooth manifold. Use a partition of unity argument to show that there is a smooth positive function $f\colon M \to \mathbb{R}$ such that $f^{-1}([0, c])$ is compact for every $c \geq 0$.

**I.4** Let $G$ be a $\Theta$–graph (see figure).

a) Explicitly compute a regular degree 3 cover of $G$ and its induced subgroup of $\pi_1(G)$.

b) Explicitly compute an irregular degree 3 cover of $G$ and its induced subgroup of $\pi_1(G)$.

You should provide justifications for why the first cover is regular and the second irregular.

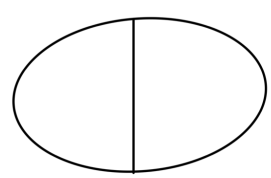

Figure 1: A $\Theta$–graph

**Part II. Solve two of the following problems.**

**II.1** Let $M$ and $N$ be surfaces of genus 2. Let $X$ be a 2–complex constructed by identifying a separating curve on $M$ with a non-separating curve on $N$ (see Figure.)

a) Compute a presentation for $\pi_1(X)$ with finitely many generators and relations.

b) Is there a retraction $X \to N$?

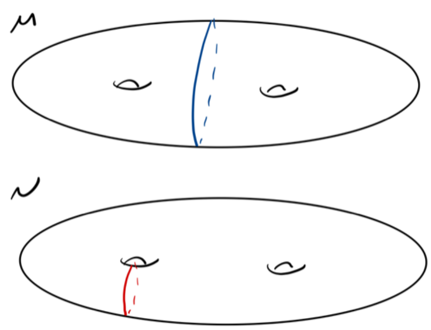


Figure 2: The surfaces $M$ with a separating curve (top) and $N$ with a non-separating curve (bottom).

**II.2** Consider the torus $T = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$T = \{(w, x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 : w^2 + x^2 = 1 = y^2 + z^2\},$$

oriented by the standard orientation on $S^1$ and the product orientation. Let $\iota\colon T \to \mathbb{R}^4$ be the inclusion map and consider the following 2-form on $\mathbb{R}^4$:

$$\eta = wxyz\, dy \wedge dw.$$

a) Is the pullback $\iota^*\eta \in \Omega^2(T)$ closed?

b) Is the pullback $\iota^*\eta \in \Omega^2(T)$ exact? (Hint: computing $\int_T \iota^*\eta$ is useful.)

**II.3** Suppose that $M, N$ are compact, connected smooth manifolds without boundary, of the same dimension. Suppose that $f\colon M \to N$ has no critical points. Show that $f_*\colon \pi_1(M) \to \pi_1(N)$ is injective. Give a counter-example without the compactness assumption.