Comprehensive Examination in Geometry & Topology
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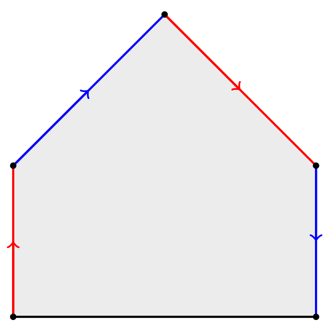
Part I. Solve three of the following problems.

I.1 Let X be the space defined by identifying two pairs of sides of a pentagon as indicated:



Compute $\pi_1(X)$, then equip X with a Δ-complex structure and use it to compute the homology groups $H_i(X)$ for all $i \geq 0$. (Note, you must use a Δ-complex structure to compute the homology groups.)

I.2 Let (X, x) be a connected CW complex with base point and let $p\colon (\widetilde{X}, \widetilde{x}) \to (X, x)$ be its universal cover with $\widetilde{x} \in p^{-1}(x)$. Identify $\pi_1(X, x)$ with the deck group of p.

Let $f\colon (X, x) \to (X, x)$ be a continuous map. Show that

1. f has a unique lift to $\widetilde{f}\colon (\widetilde{X}, \widetilde{x}) \to (\widetilde{X}, \widetilde{x})$.
2. For any $\gamma \in \pi_1(X, x)$ and $z \in \widetilde{X}$,
$$\widetilde{f}(\gamma \cdot z) = f_*(\gamma) \cdot \widetilde{f}(z),$$
where $f_*\colon \pi_1(X, x) \to \pi_1(X, x)$ is the induced map and $\gamma \cdot z$ is the image of $z \in \widetilde{X}$ under the deck transformation corresponding to $\gamma \in \pi_1(X, x)$.

I.3 Show that
$$\{(x, y, z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 - 3xyz = 1\}$$
is a surface in $\mathbb{R}^3$. Identify its tangent space at $(1, 0, 0)$.

I.4 Suppose that M is a smooth compact manifold without boundary that admits a smooth, nowhere vanishing vectorfield. Show that there is a diffeomorphism $F\colon M \to M$ that has no fixed points.

Part II. Solve two of the following problems.

II.1 Let M be a smooth submanifold of $\mathbb{R}^n$ of codimension greater than 2. Show that $\mathbb{R}^n - M$ is connected and simply connected.

II.2 Let M and N be smooth, oriented, compact manifolds of dimension n (without boundary) and suppose that $f\colon M \to N$ is smooth.

1. Show that if there is an n-form ω on N such that $\int_M f^*\omega \neq 0$, then f is surjective.
2. Show that if there is an n-form ω on N such that $\int_M f^*\omega = \int_N \omega \neq 0$, then $f\colon \pi_1(M) \to \pi_1(N)$ is surjective.

II.3 Let X be a connected topological space with compact universal cover. Prove that for each $n \geq 1$, every continuous map from X to the n-torus $T^n = (S^1)^n$ is homotopic to a constant map.