Comprehensive Examination in Geometry & Topology
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January 2019

Part I. Do three of these problems.

I.1 State and prove the Brouwer fixed point theorem in n dimensions. You may assume standard facts about homology groups of standard spaces.

I.2 Let M be a compact orientable 4–manifold without boundary. Let α be a smooth 1–form on M and let β be a smooth 2–form on M. Prove that

$$\int_M \alpha \wedge d\beta = \int_M d\alpha \wedge \beta.$$

I.3 Let $S \subset \mathbb{R}^3$ be a sphere of radius 2. Define a function $f : S \to \mathbb{R}$ via

$$f(x, y, z) = x^3 + y^3 + z^3 + 1.$$

Let $N = f^{-1}(0)$. Prove that N is a smooth submanifold of S. Then, give an explicit description of the tangent space T_pN at a point $p = (x, y, z) \in N$.

I.4 Let M be a compact orientable n–manifold without boundary.

a) Use a partition of unity to construct an everywhere–positive n–form ω on M.

b) Is ω closed?

c) Is ω exact?

Part II. Do two of these problems.

II.1 Let M be the topological space obtained from a unit cube by identifying every pair of opposite faces via a 90° clockwise twist. Let X be the 2–skeleton of M.

a) Give a presentation for $\pi_1(M)$.

b) Compute the homology groups $H_i(X)$.

c) Compute the homology groups $H_i(M)$. *Hint:* these can be obtained from the homology groups of X via the Mayer–Vietoris sequence.

II.2 Let $f : M \to N$ be a smooth submersion. Let W be a nowhere vanishing vector field on N. Construct a nowhere vanishing vector field V on M, such that $f_* V = W$.

II.3 Let $A = S^1 \times S^1$ be a torus. Let B be a Möbius band. Let X be the topological space obtained by identifying ∂B to the circle $S^1 \times \{*\} \subset A$.

a) Let $\varphi : \pi_1(X) \to \mathbb{Z}/2$ be the homomorphism that sends every loop in X to 0 and the generator of $\pi_1(B)$ to 1. Describe the cover $\hat{X} \to X$ corresponding to $\ker(\varphi)$. *Hint:* The space $\hat{X}$ embeds in $\mathbb{R}^3$. A complete answer to this question should include a fairly accurate picture.

b) Let $f : \hat{X} \to \hat{X}$ be the non-trivial deck transformation of the cover. Describe the action of f.

c) Describe the universal cover $\widetilde{X}$.