Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Let X be the space obtained from identifying sides of the polygon $\mathcal{P}$ in Figure 1 as indicated. Compute $\pi_1(X)$ and the homology groups $H_*(X;\mathbb{Z})$.

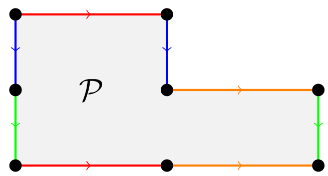

Figure 1: Identifying sides of a polygon $\mathcal{P}$.

I.2 Consider the function

$$f(x, y, z) = xy + yz + zx + x + y + z$$

from $\mathbb{R}^3$ to $\mathbb{R}$. Classify all the $t \in \mathbb{R}$ such that $M_t = f^{-1}(t)$ is a smooth 2-dimensional submanifold of $\mathbb{R}^3$.

I.3 Let K be a Klein bottle and X be the wedge product $\mathbb{RP}^2 \vee \mathbb{RP}^2$ of two real projective planes. If $f : X \to K$ is a continuous mapping, prove that f is homotopic to a constant map.

I.4 Suppose that X and Y are smooth manifolds and that $f : X \to Y$ is a smooth map with differential $(Df_x) : T_xX \to T_{f(x)}Y$.

(a) Let V be a smooth vector field on Y, with $V_y \in T_yY$ the vector in V at y. Say that V is *tangent* to f if

$$V_{f(x)} \in Df(T_xX) \subseteq T_{f(x)}Y$$

for all x in X. If f is an immersion and V is tangent to f, prove that $(Df)_x^{-1}(V_{f(x)})$ determines a well-defined smooth vector field $f^*(V)$ on X. If V is nowhere-vanishing, prove that $f^*(V)$ is also nowhere-vanishing.

(b) Give an example of X, Y, f, and a nowhere-vanishing V satisfying all the assumptions of part **(a)**.

Part II. Solve two of the following problems.

II.1 Let R_0 denote the graph with two vertices and one edge between them; see Figure 2. For $n \geq 1$, the *rose with n petals* R_n is the unique finite graph with 1 vertex and n edges; see Figure 3.

Figure 2: The graph R_0.

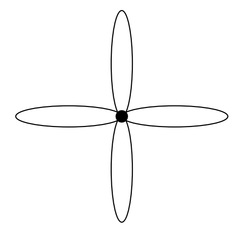

Figure 3: The graph R_4.

(a) Prove that any finite connected graph $\mathcal{G}$ is homotopy equivalent to R_n for some $n \geq 0$.

(b) Let $\mathcal{G}$ be a finite connected graph, V denote the number of vertices of $\mathcal{G}$, and E the number of edges. Give a formula for the n in part **(a)** in terms of V and E.

(c) Let $\mathcal{G}$ be as above and $\mathcal{H} \to \mathcal{G}$ be a connected covering of degree d. Show that $\pi_1(\mathcal{H})$ is a free group of rank $m \geq 0$, where m is an explicit function of: d, the number V of vertices of $\mathcal{G}$, and the number E of edges of $\mathcal{G}$.

II.2 Let M be a smooth compact oriented manifold of dimension n without boundary and $\Omega^q(M)$ be the space of smooth exterior forms of degree q on M. Prove that the assignment

$$I(\omega) := \int_M \omega$$

induces an $\mathbb{R}$-linear map

$$I : \Omega^n(M)/d(\Omega^{n-1}(M)) \longrightarrow \mathbb{R},$$

where d is the exterior derivative. Use a partition of unity to prove that the map I is onto.

II.3 Let $X = S^1$ with coordinate

$$\theta \mapsto z = \cos\theta + i\sin\theta \in \mathbb{C}$$

and $Y = X \times X$.

(a) Show that the map $f_n(z) = (z, z^n)$ defines a smooth immersion of X into Y for all $n \in \mathbb{Z}$.

(b) Give an example of a smooth map $h : Y \to X$ such that $h \circ f_1$ is a submersion at every $z \in X$.

(c) Give an example of a smooth map $h : Y \to X$ such that $h \circ f_1$ *fails* to be a submersion for every $z \in X$.