Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Prove that

$$\mathrm{O}(n) = \{A \in \mathrm{M}_n(\mathbb{R}) \ : \ {}^tAA = \mathrm{Id}\}$$

is a smooth submanifold of the space $\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$ of $n \times n$ matrices. Here, ${}^tA$ is the transpose of the matrix A. Describe a natural identification of the tangent space to $\mathrm{M}_n(\mathbb{R})$ at the identity with $\mathrm{M}_n(\mathbb{R})$, and describe the tangent space to $\mathrm{O}(n)$ at the identity as a subspace of $\mathrm{M}_n(\mathbb{R})$.

I.2 Give a careful definition of a normal covering map. Also, give

- an example of a normal covering map p that is not a homeomorphism, and
- an example of a non-normal covering map p.

For both examples, prove that the corresponding covering map is normal (resp. non-normal). *Hint: there is a non-normal covering map with the base being the Klein bottle.*

I.3 Consider the circle S^1 as the unit circle in $\mathbb{R}^2$ with coordinate θ, then give the torus $T = S^1 \times S^1$ into $\mathbb{R}^4$ coordinates (θ_1, θ_2). Define differential forms

$$\eta_i = \mathrm{d}\theta_i \quad i = 1, 2.$$

(a) Given $(a, b), (c, d) \in \mathbb{R}^2$, calculate

$$(a\eta_1 + b\eta_2) \wedge (c\eta_1 + d\eta_2).$$

(b) Calculate

$$\int_T \eta_1 \wedge \eta_2.$$

(c) Prove that $a\eta_1 + b\eta_2$ is closed, but not exact, for every $(a, b) \in \mathbb{R}^2$ except $(0, 0)$.

I.4 Let M be a closed 4-manifold and let $\mathbb{CP}^2$ be the complex projective plane. Compute the homology groups $H_*(X; \mathbb{Z})$ of the *blowup* $X = M\#\mathbb{CP}^2$ of M at a point (i.e., the connect sum of M and $\mathbb{CP}^2$) in terms of the homology groups of M.

Part II. Solve two of the following problems.

II.1 Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be a 2-torus.

(a) Let L be a line of slope (p, q) in $\mathbb{R}^2$, with p and q integers (not both zero). Prove that L projects to a homotopically nontrival closed loop $\sigma_{p,q}$ on T.

(b) Let T_1 and T_2 be two copies of T, and let X be the space given by gluing T_1 to T_2 by gluing a (p_1, q_1) curve on T_1 to a (p_2, q_2) curve on T_2. Compute $\pi_1(X)$. You do not need to prove that $\pi_1(T) \cong \mathbb{Z} \times \mathbb{Z}$.

II.2 State the classification of closed, connected 2-dimensional manifolds. For each integer $k \geq 1$, identify all such manifolds M with $H_k(M; \mathbb{Z}) = \{0\}$. Also identify all such M with $H_k(M; \mathbb{Q}) = \{0\}$.

II.3 Let X be a closed manifold and Y an embedded closed submanifold with Euler characteristic zero. Show that X and $X \setminus Y$ have the same Euler characteristic.