Comprehensive Examination in Geometry & Topology
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Part I. Do three of these problems.

I.1 Give a careful definition of an *orientation* on a manifold M. Use this definition to prove that the annulus $S^1 \times (0,1)$ is orientable, but the Möbius band is not.

I.2 In a smooth manifold M, let U be an open set and $A \subset U$ a closed set. Prove that there is a smooth function $f : M \to \mathbb{R}$, supported in U, which evaluates to 1 everywhere on A.

I.3 Let $T^n = \left(S^1\right)^n$ be the n-torus. Prove that $\chi(T^n) = 0$.

I.4 Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(a,b,c) = (a^2 + b^2 + c^2, 2a + b - c).$$

For what values of t is $X_t = f^{-1}(1,t)$ a manifold? For every t such that X_t is a manifold, calculate the dimension of X_t.

Part II. Do two of these problems.

II.1 Let $D = \{z \in \mathbb{C} : |z| \leq 1/2\}$, and let $A = \{z \in \mathbb{C} : 1 \leq |z| \leq 2\}$. Let $X = A \cup_f D$ be the topological space obtained by gluing ∂D to the inner circle of ∂A via the map $f(z) = z^7/2$.

(a) Compute $\pi_1(X)$.

(b) Prove that any map $g : X \to S^1$ is homotopic to a constant map.

II.2 Let Y_1 and Y_2 be two copies of real projective space $\mathbb{RP}^3$. Let $Z_i \subset Y_i$ be the copy of $\mathbb{RP}^1$ obtained by restricting two coordinates to 0. Let X be the space obtained from the union of Y_1 and Y_2 where their intersection is $Z_1 = Z_2$. Use the Mayer–Vietoris Theorem to calculate the homology groups of X.

II.3 A *closed curve* on a manifold X is a smooth map $\gamma : S^1 \to X$. If ω is a 1-form on X, define the *line integral* of ω around γ by

$$\oint_\gamma \omega = \int_{S^1} \gamma^* \omega.$$

(a) For the case $X = \mathbb{R}^k$, write $\oint_\gamma \omega$ explicitly in terms of the coordinate expressions of γ and ω.

(b) Define a 1-form on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ by

$$\omega(x, y) = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

Calculate $\int_C \omega$ for any circle C of radius r around the origin.