Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Let M be a compact, oriented n–manifold without boundary. Let ω be a smooth $(n-1)$ form on M. Show that there is some $p \in M$ such that $(d\omega)_p = 0$.

I.2 Let S be the unit 2–sphere in $\mathbb{R}^3$, centered at the origin. For $t \in \mathbb{R}$, let P_t be the surface defined by the equation $y^2 + z = t$.

Determine the values of $t \in \mathbb{R}$ for which P_t is transverse to S, and identify the dimension of $P_t \cap S$ at these values.

I.3 Prove that the n–sphere S^n admits a nonvanishing vector field if and only if n is odd.

I.4 Let X be the space formed by quotienting the disk D^2 by the following relation

$$z \sim w \iff z = w \text{ or } w = e^{2\pi/7} z \text{ for } w, z \in \partial D^2.$$

a) Compute $\pi_1(X)$ and describe a cell structure on $\widetilde{X}$.

b) Does X have a connected double cover? Why or why not?

c) Let $f\colon X \to Y$ be a continuous map where Y is the torus. Show that f is homotopic to a constant map.

Part II. Solve two of the following problems.

II.1 Let S be a closed, orientable surface of genus 2. Consider the space X obtained by attaching a cylinder to S by mapping one of its boundary components by homeomorphism to an essential, separating curve of S and its other boundary component by homeomorphism to a nonseparating curve of S (see Figure 1). Compute $H_k(X)$ for all $k \geq 0$.

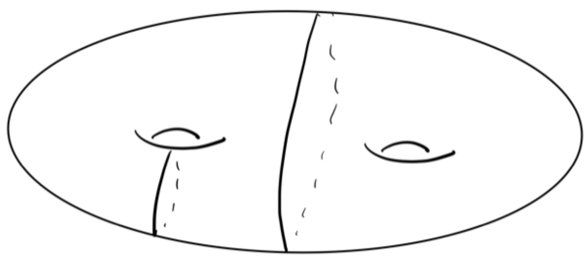

Figure 1: Curves attached to the annulus

II.2 Let M be a smooth n–manifold and $f\colon M \to \mathbb{R}^N$ be a smooth embedding.

a) Prove that f induces a smooth embedding $F\colon TM \to T\mathbb{R}^N$.

b) Let $UM \subset TM$ be the subset consisting of vectors v such that $f_*(v)$ has unit length. Prove that UM is an embedded submanifold of dimension $2n-1$.

II.3 Let $T = S^1 \times S^1$ be a torus. What the minimum number of cells that need to be attached to T to make it contractible? Prove your answer.