Comprehensive Examination in Geometry & Topology
Department of Mathematics, Temple University

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Part I. Solve three of the following problems.

I.1 Let $a \neq b$ be non-zero real numbers and $\varepsilon \in \mathbb{R}$. Consider the subset of points in $\mathbb{R}^2$:

$$X_\varepsilon := \{(x, y) \ : \ y^2 = x(x - \varepsilon a)(x - \varepsilon b)\}.$$

a) Prove that, if $\varepsilon \neq 0$, then X_ε is a smooth submanifold of $\mathbb{R}^2$.

b) Prove that, as the space with the subset topology, X_0 admits a smooth structure such that X_0 is diffeomorphic to $\mathbb{R}$.

c) Show that X_0 is not a smooth submanifold of $\mathbb{R}^2$.

I.2 Let M be a smooth, connected manifold such that $H_1(M) \cong \mathbb{Z}/5\mathbb{Z}$. Must M be orientable? Either prove that the answer is "yes," or construct a counterexample. *Hint:* Covering space theory is relevant to this question.

I.3 Construct a Δ–complex X whose homology groups are as follows:

$$H_k(X) = \begin{cases} \mathbb{Z}^2 & k = 0 \\ \mathbb{Z}^2 \oplus \mathbb{Z}/3\mathbb{Z} & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & k > 2 \end{cases}$$

I.4 Let S_g be the oriented surface of genus g, embedded in $\mathbb{R}^3$ in the standard way. Here, "in a standard way" means that the region inside S_g is an ϵ–neighborhood of a graph. Let V_g be the handlebody of genus g: that is, the union of S_g and the space inside it. Compute the relative homology groups $H_i(V_g, S_g)$.

Part II. Solve two of the following problems.

II.1 Let $X = S \cup I$ be the union of an orientable genus 2 surface S and an interval I, which is attached to S at its endpoints as shown in the figure:

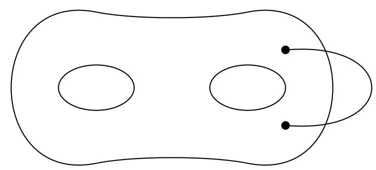

Figure 1: The space $X := S \cup I$

a) Compute $\pi_1(X)$.

b) Construct a connected 3-fold covering map $f : Y \to X$ where $f^{-1}(S)$ is connected. Sketch a picture of Y.

c) Construct a connected 3-fold covering map $h : Z \to X$ where $h^{-1}(S)$ is disconnected. Sketch a picture of Z.

II.2 Let M be a smooth manifold (with or without boundary), and let $A \subset M$ be an arbitrary non-empty subset. Let $f : A \to \mathbb{R}^k$ be a smooth function.

a) Give a careful definition of what it means for $f : A \to \mathbb{R}^k$ to be smooth, without making any additional assumptions about the subset A.

b) Let A be a closed subset of M, and let U be an open subset of M containing A. Prove that there exists a smooth function $\tilde{f} : M \to \mathbb{R}^k$ such that

$$\tilde{f}\big|_A = f \qquad \text{and} \qquad \operatorname{supp}(\tilde{f}) \subset U.$$

Hint: use partitions of unity.

c) Give a counterexample to show that the above conclusion about the existence of $\tilde{f}$ can be false if A is not closed.

II.3 Consider S^2 with the following two-chart atlas: the complement U_N of the north pole is mapped onto $\mathbb{R}^2$ with coordinates (x, y) and the complement U_S of the south pole is mapped onto another copy of $\mathbb{R}^2$ with coordinates $(\tilde{x}, \tilde{y})$. The transition function on the overlap is defined by

$$\tilde{x} := \frac{x}{x^2 + y^2}, \qquad \tilde{y} := -\frac{y}{x^2 + y^2} .$$

Consider the 2-form ω, defined in coordinates by the formulas

$$\omega_N := \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}, \qquad \omega_S := \frac{d\tilde{x} \wedge d\tilde{y}}{(1 + \tilde{x}^2 + \tilde{y}^2)^2}. \tag{1}$$

You do not need to verify that ω_N and ω_S agree on the overlap.

a) Prove that the 2-form ω is closed but not exact.

b) Let $V \subset U_N$ be the open subset corresponding to the upper half plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$. Find a smooth 1-form ψ on V such that $\omega\big|_V = d\psi$.