Comprehensive Examination in Geometry & Topology
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August 2021

Part I. Solve three of the following problems.

I.1 Let M, N be smooth manifolds and let $f\colon M \to N$ be a smooth map.

a) Define what it means for a subset $Y \subset N$ to have measure zero.

b) Formulate the Morse–Sard theorem for f (also known as Sard's Theorem). *You do not have to prove it.*

c) Prove that if $\dim(M) < \dim(N)$, then the image $f(M)$ has measure zero in N. You may use standard facts about Lebesgue measure in $\mathbb{R}^n$, but you may *not* use the Morse–Sard theorem itself.

I.2 Let $n \in \mathbb{Z}_{\geq 2}$ and $\vec{u}$ be a non-zero vector in $\mathbb{R}^n$. Denote by $X_{\vec{u}}$ be the following subset of $\mathrm{GL}_n(\mathbb{R})$:

$$X_{\vec{u}} := \{A \in \mathrm{GL}_n(\mathbb{R}) \mid A\vec{u} = \vec{u}\}.$$

Prove that $X_{\vec{u}}$ is a submanifold of $\mathrm{GL}_n(\mathbb{R})$ diffeomorphic to $\mathrm{GL}_{n-1}(\mathbb{R}) \times \mathbb{R}^{n-1}$.

I.3 Let M be a compact connected n–manifold without boundary, where n is odd. Show that the Euler characteristic of M is zero. *Hint:* Use Poincaré duality.

I.4 The following are two parts of the same problem.

a) Show that the fundamental group of a finite, connected graph is a free group of finite rank.

b) Let G be a finite group with d elements, and let $\phi\colon F_n \to G$ be a surjective map from a free group of rank n. Show that $\ker(\phi)$ is a free group and compute its rank. *Hint:* Use part (a).

Part II. Solve two of the following problems.

II.1 Let M be a smooth manifold of dimension k without boundary and $f\colon M \to \mathbb{R}^n$ be a smooth immersion. Recall that the Grassmanian $G_k(\mathbb{R}^n)$ is the space of k–dimensional subspaces of $\mathbb{R}^n$. We assume that $k < n$.

a) Prove that the formula
$$g(p) := df_p(T_pM), \qquad p \in M$$
defines a smooth map $g\colon M \to G_k(\mathbb{R}^n)$ to the Grassmanian.

b) Let $f\colon S^{n-1} \to \mathbb{R}^n$ be the standard inclusion map of the unit sphere S^{n-1} into $\mathbb{R}^n$. Show that the corresponding map $g : S^{n-1} \to G_{n-1}(\mathbb{R}^n)$ is onto but it is not 1-1.

II.2 Let X be the space obtained from two n-spheres by identifying them along their equatorial $(n-1)$-sphere. Using any method, compute the homology groups $H_i(X)$ for all i. The homology groups of an n-sphere can be used without proof.

II.3 Let V be a continuous vector field on the unit ball $B^n \subset \mathbb{R}^n$ which is nowhere zero. Prove that there are points $x, y \in \partial B^n \cong S^{n-1}$ and positive numbers a, b such that $V(x) = ax$ and $V(y) = -by$.