Comprehensive Exam in Geometry & Topology

Temple University

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Part I: Do three of the following problems.

1. Suppose $f\colon M \to N$ is a smooth map between smooth manifolds and $S \subset N$ is a smooth submanifold. Suppose that for every $p \in f^{-1}(S)$, the vector spaces $f_*(T_pM)$ and $T_{f(p)}S$ span $T_{f(p)}N$. Prove that $f^{-1}(S)$ is a smooth submanifold of M, and compute its dimension.

2. Let $X = T^2 \vee S^1$. Compute $\pi_1(X)$, draw a picture of the universal cover $\widetilde{X}$, and explain how $\pi_1(X)$ acts on $\widetilde{X}$.

3. Let X be a connected CW complex such that $H_1(X) \cong \mathbb{Z}/3$.
(a) Does X have a connected 2–fold cover? Prove your answer.
(b) Does X have a connected 3–fold cover? Prove your answer.

4. Let M be a non-orientable manifold and B^k a ball. Prove that $M \times B^k$ is non-orientable.

Part II: Do two of the following problems.

1. Let M be a closed, orientable, smooth manifold. Let $S \subset M$ be an orientable submanifold of codimension 1. Construct a smooth flow $\varphi_t\colon M \to M$ that takes S off itself. That is, construct φ_t and show there is an $\epsilon > 0$ such that $\varphi_t(S) \cap S = \emptyset$ for $t \in (0, \epsilon)$.

2. This problem is about the reduced homology groups of $\mathbb{RP}^n$.
(a) Prove that

$$\widetilde{H}_n(\mathbb{RP}^n) = \begin{cases} 0, & n \text{ is even} \\ \mathbb{Z}, & n \text{ is odd.} \end{cases}$$

(b) Use the long exact sequence of the pair $(\mathbb{RP}^n, \mathbb{RP}^{n-1})$ to show that

$$\widetilde{H}_{n-1}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

(c) Use induction to compute $\widetilde{H}_k(\mathbb{RP}^n)$ for all k. You may assume the answer to parts (a), (b) in addition to standard facts about homology groups of standard spaces.

3. Consider the torus $T = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2$ given by

$$T = \{(w, x, y, z) \in \mathbb{R}^2 \times \mathbb{R}^2 : w^2 + x^2 = 1 = y^2 + z^2\},$$

Consider the 2-form $\omega = xyz\, dw \wedge dy$, restricted to T.

(a) Is ω closed? Justify your answer.

(b) Is ω exact? Justify your answer. *Hint:* integrating ω over T is helpful.