Comprehensive Examination in Geometry & Topology
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Part I. Solve three of the following problems.

I.1 Define a simply connected space and prove that S^n is simply connected for every $n \geq 2$.

I.2 Define the manifold $\mathbb{RP}^n$ and describe a CW structure on it. Use this CW structure to compute $H_\bullet(\mathbb{RP}^n, \mathbb{Z})$ and deduce that $\mathbb{RP}^n$ is not orientable if n is even.

I.3 Let

$$\pi : \mathbb{R}^3 \smallsetminus \{(0,0,0)\} \to \mathbb{RP}^2$$

be the usual projection. Show that the nonzero solutions to $x^2 + y^2 = z^2$ are of the form $\pi^{-1}(X)$ for some $X \subset \mathbb{RP}^2$. Show that X is a submanifold.

I.4 Let $j : S^2 \to \mathbb{R}^3$ be the embedding of the standard 2-dimensional sphere

$$S^2 = \{\vec{x} = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

into $\mathbb{R}^3$ and ω be the following 2-form on $\mathbb{R}^3$:

$$\omega = x_1 dx_2 dx_3 - x_2 dx_1 dx_3 + x_3 dx_1 dx_2.$$

Prove that the pullback $j^*\omega$ is a closed form on S^2 that is not exact.

Part II. Solve two of the following problems.

II.1 Consider the space X obtained from the 2-torus $T^2 = S^1 \times S^1$ by removing a small open disc and identifying antipodal points of the resulting boundary. (See figure 1.)

(a) Compute $\pi_1(X)$.

(b) Find a Δ-complex structure on X and compute $H_1(X, \mathbb{Z})$ using this Δ-complex structure.

(c) Verify your answer using the relationship between $\pi_1(X)$ and $H_1(X, \mathbb{Z})$.

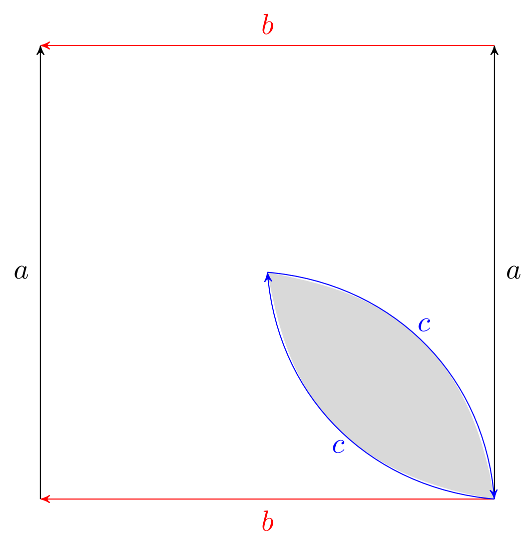

Figure 1: The gray area is removed. Letters indicate how the segments are glued to each other.

II.2 Let n be an integer $n \geq 2$ and X_n be the space that consists of n two-dimensional discs $D_1, D_2, \ldots, D_n$ with their boundary circles identified. Let $Y_n := S^1 \sqcup_f D^2$, where

$$f(z) = z^n.$$

(a) Prove that $\pi_1(Y_n) = \mathbb{Z}/n\mathbb{Z}$ and X_n is the universal covering space for Y_n.

(b) Use X_6 to describe all isomorphism classes of path-connected covering spaces of Y_6.

II.3 Let $M \subset \mathbb{R}^6$ be the space of distinct points on the 2-sphere S^2, i.e.,

$$M = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \ : \ x, y \in S^2, \ x \neq y\}.$$

(a) Prove that M is a manifold and calculate its dimension.

(b) Consider the function $f : M \to \mathbb{R}$ given by $f(x, y) = |x - y|^2$, where $|\ |$ is the Euclidean distance on $\mathbb{R}^3$. Prove that f is smooth and that $1/2$ is a regular value. What is $\dim(f^{-1}(1/2))$?

(c) Give a topological description of $\dim(f^{-1}(1/2))$.