Ph.D. Comprehensive Examination
Differential Geometry and Topology

August 2014

Part I. Do three of these problems.

I.1 Let Y be a torus, D_0 a closed disk in Y, let D be the closed disk in $\mathbb{R}^2$, and let X be the result of gluing the boundary of D to the boundary of D_0.

a) Use van Kampen's Theorem to find $\pi_1(X)$.

b) Use the Mayer-Vietoris Theorem to find $H_k(X)$.

I.2 Let X be a smooth manifold. Give a careful definition of its tangent bundle TX. Then prove that $T(\mathbb{RP}^2)$ is not diffeomorphic to $\mathbb{RP}^2 \times \mathbb{R}^2$.

I.3 Let X and Y be smooth connected manifolds (without boundary) of the same dimension with X compact. Suppose $f : X \to Y$ is an immersion (that is, $df_x : T_xX \to T_{f(x)}Y$ is injective for every x). Show that f is onto.

I.4 Prove that S^n admits a non-vanishing continuous vector field if and only if n is odd.

Part II. Do two of these problems.

II.1 Let $X = S^2 \cup A$ be the union of a 2-sphere and an axis connecting the north and south poles.

a) Compute $\pi_1(X)$.

b) Explain why X has a universal covering $\tilde{X}$, describe it (including a local picture), and describe the action of $\pi_1(X)$ on $\tilde{X}$.

c) Why does X admit an n-sheeted covering for each $n \in \mathbb{N}$?

II.2 Let X and Y be smooth manifolds, $f : X \to \mathbb{R}$ and $g : X \to Y$ smooth with g a submersion (that is, $dg_x : T_xX \to T_{g(x)}Y$ is onto for every x).

a) Let $y_0 \in g(X)$ and let $Z = g^{-1}(y_0)$. Show that Z is a smooth manifold.

b) Let $h : Z \to \mathbb{R}$ be the restriction of f to Z. Show that x_0 is a critical point of h (that is, $dh(x_0) = 0$) if and only if $df(x_0)$ belongs to the image of

$$dg^*_{x_0} : T^*_{g(x_0)}Y \to T^*_{x_0}X.$$

Remark: This is what is behind the method of Lagrange multipliers.

II.3 Let X be a smooth closed (compact without boundary) orientable n-dimensional manifold, and pick an orientation. Let $\Omega^k(X) = C^\infty(X; \bigwedge^k X)$ be the space of smooth k-forms on X and denote by $H^k_{\mathrm{dR}}(X)$ the k-th de Rham cohomology group.

a) Explain why integration of smooth n-forms is well defined.

Define $\beta : \Omega^k(X) \times \Omega^{n-k}(X) \to \mathbb{R}$ by

$$\beta(\phi, \psi) = \int_X \phi \wedge \psi.$$

b) Let $\phi \in \Omega^k(X)$. Show that

$$\beta(\phi, d\chi) = 0 \text{ for all } \chi \in \Omega^{n-k-1}(X) \iff \phi \text{ is closed.}$$

c) Show that β determines a map

$$H^k_{\mathrm{dR}}(X) \times H^{n-k}_{\mathrm{dR}}(X) \to \mathbb{R}.$$

Hint: Part c) only needs the easier direction of part b).