# Comprehensive Examination in Algebra
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## Part I. Do three of these problems.

**I.1** Let $G$ be a finite non-Abelian group that is generated by two distinct elements of order 2. Show that $G$ is isomorphic to the dihedral group $D_n := \langle r, s \mid s^2 = 1 = r^n,\ sr = r^{-1}s\rangle$ of order $2n$ for some $n \in \mathbb{Z}_{\geq 3}$.

**I.2** Let $A$ and $B$ be rings (with identity) and let $R = A \times B$ be their direct product. Prove:

(a) If $I, J$ are (two-sided) ideals of $A$ and $B$, respectively, then $I \times J$ is an ideal of $R$ and $R/(I \times J) \cong (A/I) \times (B/J)$.

(b) Every ideal of $R$ is of the form $I \times J$ for suitable ideals $I \subseteq A$ and $J \subseteq B$.

**I.3** Let $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the $\mathbb{R}$-vector space of all functions $\mathbb{R} \to \mathbb{R}$, with the usual addition and multiplication by scalars. For any subset $I \subseteq \mathbb{R}$, let $f_I \in V$ denote the characteristic function of $I$, defined by $f_I(x) = 1$ if $x \in I$ and $f_I(x) = 0$ otherwise.

(a) Let $I_\lambda$ $(\lambda \in \Lambda)$ be any family of subsets of $\mathbb{R}$ such that $I_\lambda \nsubseteq \bigcup_{\mu \neq \lambda} I_\mu$ for all $\lambda$. Prove that the functions $f_{I_\lambda}$ $(\lambda \in \Lambda)$ are linearly independent. Conclude that $\dim_{\mathbb{R}} V = \infty$.

(b) Do the characteristic functions $f_I$ $(I \subseteq \mathbb{R})$ generate the vector space $V$?

**I.4** Let $F$ be a field and let $R = F + x^2F[x]$, the set of all polynomials in the polynomial ring $F[x]$ such that the coefficient of $x$ is 0.

(a) Show that $R$ is a subring of $F[x]$.

(b) Determine the group $R^\times$ of invertible elements of $R$.

(c) Show that $R$ is not a UFD.

*Hint for (c):* prove that $x^2$ and $x^3$ are irreducible elements of $R$.

**Part II. Do two of these problems.**

**II.1** (a) Let $\mathsf{F}_n$ be the free group on $n$ generators and let $[\mathsf{F}_n, \mathsf{F}_n]$ be the subgroup of $\mathsf{F}_n$ generated by group commutators of all elements of $\mathsf{F}_n$. Show that $\mathsf{F}_n/[\mathsf{F}_n, \mathsf{F}_n] \cong \mathbb{Z}^n$.

(b) Prove that if $\mathsf{F}_n$ is isomorphic to $\mathsf{F}_m$, then $n = m$.

**II.2** Let $R$ be a ring (with identity) and let $S = \mathrm{Mat}_n(R)$ be the $n \times n$ matrix ring. View $R$ as a subring of $S$ by identifying $r \in R$ with the "scalar" matrix $\sum_{i=1}^n re_{i,i} \in S$, where $e_{i,j} \in S$ denotes the matrix having 1 in position $(i, j)$ and 0s elsewhere. Let $M$ be a left $S$-module and put $M_i = e_{i,i}M$. Show:

(a) Each $M_i$ is an $R$-submodule of $M$ and $M = M_1 \oplus \cdots \oplus M_n$ (as $R$-modules).

(b) $M_i = e_{i,j}M$ for any $j$ and $M_i \cong M_1$ as $R$-modules.

(c) $M$ is irreducible as $S$-module if and only if $M_1$ is irreducible as $R$-module. (Recall that a module is called *irreducible* if it is nonzero and the only submodules are 0 and itself.)

**II.3** Let $F$ be a number field, that is, a finite extension of $\mathbb{Q}$. Show that there are only finitely many roots of unity in $F$.