Comprehensive Examination in Algebra
Department of Mathematics, Temple University

January 2019

Part I. Do three of these problems.

I.1 (a) Give an example of an abelian group A that is both infinite and torsion (i.e., every elementy of A has finite order).

(b) Show that an infinite torsion abelian group cannot be finitely generated.

(c) Give an example of an abelian torsion group A and a subgroup $1 \neq B \leq A$ such that $A/B \cong A$.

I.2 Recall that the Cayley-Hamilton Theorem holds for matrices in $\mathrm{Mat}_n(R)$, the ring of $n \times n$ matrices over the commutative ring R. Use this to show that $AB = 1_{n\times n}$ implies $BA = 1_{n\times n}$ for $A, B \in \mathrm{Mat}_n(R)$.

I.3 Prove that the rings $\mathbb{Z}[x]$ and $\mathbb{Z}[x, y]$ are not isomorphic.

I.4 Let M, V be left modules over a ring R (not necessarily commutative) and let $\phi\colon M \to V$ and $\psi\colon V \to M$ be R-module maps such that $\phi \circ \psi = \mathrm{Id}_V$. Show: if the module M is finitely generated, then so are V and $\operatorname{Ker}\phi$.

Part II. Do two of these problems.

II.1 Let $g \in \mathrm{SL}_2(\mathbb{Z})$ have order $|g| = t < \infty$.

(a) Regarding g as an element of $\mathrm{Mat}_2(\mathbb{C})$, the set of 2×2 matrices over $\mathbb{C}$, specify the possible Jordan Canonical forms for g that may occur. Conclude that $t \in \{1, 2, 3, 4, 6\}$.

(b) Show that all possible values of t in (a) do in fact occur.

II.2 Let R be a commutative ring having pairwise distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_t$ such that $\mathfrak{m}_1^{n_1}\mathfrak{m}_2^{n_2} \ldots \mathfrak{m}_t^{n_t} = 0$ for suitable positive integers n_i. Show:

(a) $R \cong R_1 \times R_2 \times \cdots \times R_t$, where $R_i := R/\mathfrak{m}_i^{n_i}$;

(b) Elements of R_i are either nilpotent or a unit.

II.3 For given primes p and d (not necessarily distinct), determine the number of monic irreducible polynomials of degree d over the field $\mathbb{F}_p$ with p elements.