Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a finitely generated group and $H \leq G$ be a subgroup of finite index. Show that H is finitely generated.

Hint: Fix a family $(g_i)_1^r$ of generators of G that is closed under inverses and let $(x_j)_1^s$ be representatives of the distinct left cosets of H, with $x_1 = 1$. Note that each product $g_i x_j$ can be uniquely written in the form $x_{k_{ij}} h_{ij}$ for $h_{ij} \in H$.

I.2 Let R be a unique factorization domain and let F denote the field of fractions of R. Let $x \in F$ be such that $x^n = r_{n-1}x^{n-1} + r_{n-2}x^{n-2} + \cdots + r_0$ for some positive integer n and suitable $r_i \in R$. Show that $x \in R$.

I.3 Let V be a vector space over a field F. For any non-negative integer k, let $\mathcal{G}_k$ denote the collection of all invertible linear transformations $\phi\colon V \to V$ such that $\operatorname{rank}(\mathrm{Id}_V - \phi) \leq k$. Show that each $\mathcal{G}_k$ is closed under taking inverses and that $\mathcal{G}_k \mathcal{G}_l \subseteq \mathcal{G}_{k+l}$.

I.4 Let R be a left noetherian ring, M a finitely generated left R-module, and $\alpha\colon M \twoheadrightarrow M$ a surjective endomorphism of M. Prove that α is in fact an automorphism.

Hint: Consider the chain $\operatorname{Ker} \alpha \subseteq \operatorname{Ker} \alpha^2 \subseteq \operatorname{Ker} \alpha^3 \subseteq \ldots$.

Part II. Do two of these problems.

II.1 Let n be a positive integer and $\mathrm{GL}_n(\mathbb{Z})$ be the group of invertible $n \times n$-matrices with entries in $\mathbb{Z}$. Prove that, for every non-identity element $g \in \mathrm{GL}_n(\mathbb{Z})$, there exists a group homomorphism φ from $\mathrm{GL}_n(\mathbb{Z})$ to a finite group G such that $\varphi(g) \neq 1_G$.

II.2 Let G be a group, not necessarily finite, and let $A, B \leq G$ be finite-index subgroups such that $|G : A|$ and $|G : B|$ are relatively prime. Show that $G = AB$.

II.3 (a) Let $f \in \mathbb{Z}[x]$ be a monic polynomial and let $f_p \in \mathbb{F}_p[x]$ denote its reduction modulo p. Show that if f_p is irreducible for some p, then f is irreducible as well.

(b) Let $f = \Phi_8 \in \mathbb{Z}[x]$ denote the 8-th cyclotomic polynomial. Show that $f_p \in \mathbb{F}_p[x]$ is reducible for all primes p.

Hint: Note that the group $(\mathbb{Z}/8\mathbb{Z})^\times$ *is not cyclic.*