Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group. Assume that G is generated by two subgroups, N and A, such that A is abelian and N is finite and normal in G. Show that the center $Z(G)$ has finite index in G.

I.2 Let R be a unique factorization domain and let P be a nonzero prime ideal of R. Show that the following are equivalent:

(i) there is no prime ideal Q with $0 \subsetneqq Q \subsetneqq P$;

(ii) P is principal, that is, $P = Rx$ for some $x \in R$.

I.3 Let $V = \mathbb{R}^n$, where n is a positive integer, and let $v \in V$ be such that $v \cdot v = 2$, where $\cdot$ is the ordinary dot product.

(a) Show that, defining $s(x) = x - (x \cdot v)v$ for $x \in V$, one obtains a linear transformation $s = s_v \in \mathrm{End}_{\mathbb{R}}(V)$ satisfying $s^2 = \mathrm{Id}_V$ and $\mathrm{rank}(s - \mathrm{Id}_V) = 1$.

(b) Assume that $t \in \mathrm{End}_{\mathbb{R}}(V)$ satisfies $t(v) = -v$, t induces the identity on $V/\mathbb{R}v$, and the composite $f = s \circ t$ satisfies $f^m = \mathrm{Id}_V$ for some positive integer m. Show that $t = s$.

I.4 Let R be a commutative ring. Recall that an element $r \in R$ is said to be *nilpotent* if $r^n = 0$ for some integer $n \geq 0$. Prove that:

(a) The set $N = \{\text{all nilpotent elements of } R\}$ is an ideal of R that is contained in every prime ideal of R.

(b) A polynomial $f = r_0 + r_1 x + \cdots + r_d x^d \in R[x]$ is nilpotent if and only if all coefficients $r_i \in R$ are nilpotent.

Part II. Do two of these problems.

II.1 Let F_n be the group freely generated by symbols $a_1, \ldots, a_n$ and let B_n be the group with the presentation

$$B_n = \langle\, \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } \; |i-j| > 1, \;\; \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \;\; \forall\, i \,\rangle$$

Prove that the formulas

$$\Psi(\sigma_i)(a_j) = \begin{cases} a_i a_{i+1} a_i^{-1} & \text{if } \; j = i \\ a_i & \text{if } \; j = i+1 \\ a_j & \text{otherwise} \end{cases}$$

define a group homomorphism $\Psi : B_n \to \mathrm{Aut}(F_n)$, where $\mathrm{Aut}(F_n)$ denotes the automorphism group of F_n. Prove that, for every $1 \leq i \leq n-1$, the automorphism $\Psi(\sigma_i)$ has an infinite order.

II.2 Let $f(x) \in F[x]$ be a (non-constant) polynomial, where F is any field. Define the discriminant δ of $f(x)$ and prove that $\delta \in F$.

II.3 (a) Let $p(x)$ be a cubic irreducible polynomial in $\mathbb{Q}[x]$, δ be its discriminant and G be its Galois group. Prove that G is isomorphic to A_3 if and only if $\delta = q^2$ for some $q \in \mathbb{Q}$. Prove that, otherwise, $G \cong S_3$.

(b) Let $p(x) \in \mathbb{Q}[x]$ be an irreducible polynomial with at least one real root and at least one complex root with a non-zero imaginary part. Prove that the Galois group of $p(x)$ is non-Abelian.