Comprehensive Examination in Algebra
Department of Mathematics, Temple University

January 2016

Part I. Do three of these problems.

I.1 Let p and q be primes such that p divides $q-1$. Prove that there are exactly two isomorphism classes of groups of order pq. [You may use without proof the fact that $\mathrm{Aut}(Z_q) \cong Z_{q-1}$, where Z_m denotes the cyclic group of order m.]

I.2 Give a careful definition of a unique factorization domain (UFD) and a careful definition of a Euclidean domain.

(a) Prove that every Euclidean domain R is a principal ideal domain (PID).

(b) Give an example of a UFD R that is not a PID. [Do not forget to prove that your R is a UFD but not a PID.]

(c) Prove that $\mathbb{Z}[\sqrt{-5}\,]$ is an integral domain that is not a UFD.

I.3 Consider the vectors $v_1 = (x, 1, 1), v_2 = (1, x, 1), v_3 = (1, 1, x) \in \mathbb{R}^3$.

(a) Determine all real numbers x such that v_1, v_2, v_3 do *not* form a basis of $\mathbb{R}^3$

(b) For each x found in (a), determine the dimension of the subspace spanned by v_1, v_2, v_3 and a basis for the orthogonal complement $\{v \in \mathbb{R}^3 \mid v \cdot v_i = 0 \text{ for } i = 1, 2, 3\}$.

I.4 Let $F[x_1, \ldots, x_n]$ be the polynomial algebra in n variables over the field F and let R be the ring of all functions $F^n \to F$, with the usual "pointwise" addition and multiplication of functions: $(f+g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ and $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for $\mathbf{x} \in F^n$. Consider the map $\phi\colon F[x_1, \ldots, x_n] \to R$ given by $\phi(f)(\mathbf{x}) = f(\xi_1, \ldots, \xi_n)$ for $\mathbf{x} = (\xi_1, \ldots, \xi_n) \in F^n$.

(a) Show that ϕ is injective if and only if F is infinite.

(b) Assume that F is finite and let $n = 1$. Show that ϕ is surjective and determine the kernel of ϕ. [You do not have to prove the standard fact about the determinant of the Vandermonde matrix.]

Part II. Do two of these problems.

II.1 Let $\mathbb{F}_3$ be the field with 3 elements and let $G := \mathrm{SL}_2(\mathbb{F}_3)$ be the group of 2×2-matrices of determinant 1 with entries in $\mathbb{F}_3$.

(a) Prove that the order of G is 24.

(b) Find a Sylow 3-subgroup and a Sylow 2-subgroup of G.

(c) Prove that G is not a simple group.

II.2 Let $\mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[x])$ be the set of all $\mathbb{Q}$-linear maps from $\mathbb{Q}[x]$ to itself, viewed as a ring with "pointwise" addition and with multiplication given by composition:

$$(A+B)\big(f(x)\big) := A\big(f(x)\big) + B\big(f(x)\big) \quad \text{and} \quad (AB)\big(f(x)\big) := A\big(B(f(x))\big)$$

for $A, B \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[x])$ and $f(x) \in \mathbb{Q}[x]$. Let Diff be the subring[1] of $\mathrm{End}_{\mathbb{Q}}(\mathbb{Q}[x])$ that is generated by the maps

$$f(x) \mapsto \lambda f(x) \quad (\lambda \in \mathbb{Q}), \qquad f(x) \mapsto x f(x), \qquad \text{and} \qquad f(x) \mapsto \frac{d}{dx} f(x).$$

(a) Prove that the assignment

$$a_0(x) + a_1(x)p + \cdots + a_n(x)p^n \longmapsto a_0(x) + a_1(x)\frac{d}{dx} + \cdots + a_n(x)\frac{d^n}{dx^n}$$

is an isomorphism of $\mathbb{Q}$-vector spaces $\mathbb{Q}[x,p] \cong \mathrm{Diff}$.

(b) Prove that an element $A \in \mathrm{Diff}$ belongs to the center of Diff if and only if there exists $\lambda \in \mathbb{Q}$ such that $A(f(x)) = \lambda f(x)$ for all $f(x) \in \mathbb{Q}[x]$.

II.3 Let $\zeta_n := e^{\frac{2\pi i}{n}} \in \mathbb{C}$.

(a) Determine the minimal polynomial of ζ_8 over $\mathbb{Q}(\sqrt{2})$.

(b) Determine the minimal polynomial of ζ_{120} over $\mathbb{Q}$.

[1] Diff is called the ring (or algebra) of differential operators on the affine line over $\mathbb{Q}$.