Comprehensive Examination in Algebra
Department of Mathematics, Temple University

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Part I. Do three of these problems.

I.1 Let G be a finite group with the property that all of its Sylow subgroups are normal. Further suppose, for all prime numbers p, that p^2 does not divide $|G|$. Prove that G is abelian.

I.2 Let R be a ring with multiplicative identity, and suppose that R is irreducible as left R-module. Prove that R is a division ring. (Note: It is not sufficient to show that every nonzero element of R has a left multiplicative inverse.)

I.3 Let V be a finite-dimensional vector space over the complex field $\mathbb{C}$ and let $\phi \in \operatorname{End}_{\mathbb{C}}(V)$ be a linear operator on V. For each $\lambda \in \mathbb{C}$, put

$$V^{\lambda} := \left\{ v \in V \mid (\phi - \lambda)^t(v) = 0 \text{ for some } t \geq 0 \right\}$$

Prove:

(a) Each V^{λ} is a subspace of V.

(b) $V^{\lambda} \neq 0$ if and only if λ is an eigenvalue of ϕ.

(b) V is the direct sum of the nonzero subspaces V^{λ} $(\lambda \in \mathbb{C})$.

I.4 Prove that the additive group $\mathbb{Q}$ of rational numbers has no subgroups of finite index other than $\mathbb{Q}$ itself.

Part II. Do two of these problems.

II.1 Let G be a group (not necessarily finite).

(a) Show that G is abelian if and only if there exists a collection of normal subgroups N_i $(i \in I)$ such that $\bigcap_{i\in I} N_i = \{1\}$ and all G/N_i are abelian.

(b) Does the assertion of (a) remain true if "abelian" is replaced by "finite"? Please give a proof or a counterexample.

(c) Does the assertion of (a) remain true if "abelian" is replaced by "nilpotent"? Please give a proof or a counterexample.

II.2 Consider the polynomial algebra $\mathbb{R}[x]$ over the field $\mathbb{R}$. For each pair of integers $i, j \geq 0$, let $\phi_{i,j}$ be the $\mathbb{R}$-linear operator on $\mathbb{R}[x]$ that is given by

$$\phi_{i,j}\big(f(x)\big) = x^i \tfrac{d^j}{dx^j} f(x)$$

Prove:

(a) The operators $\phi_{i,j}$ are linearly independent over $\mathbb{R}$.

(b) The only subspaces of $\mathbb{R}[x]$ that are stable under all $\phi_{i,j}$ are $\{0\}$ and $\mathbb{R}[x]$.

II.3 Let $\zeta = e^{2\pi i/11} \in \mathbb{C}$.

(a) Show that $\alpha = \zeta + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^9$ generates a field of degree 2 over $\mathbb{Q}$ and find the minimal polynomial of α over $\mathbb{Q}$.

(b) Find an element $\beta \in \mathbb{Q}(\zeta)$ such that $[\mathbb{Q}(\beta) : \mathbb{Q}] = 5$ and find the minimal polynomial of β over $\mathbb{Q}$.

Part III. An alternate for possible inclusion in Part I

III.1 Prove that $\mathbb{Z}$ and $\mathbb{Z}[x]$ are not isomorphic as rings.