Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1. Let G be a group containing only finitely many subgroups (including the non-normal subgroups). Prove that G is a finite group.

I.2. Let R be a ring with identity 1. Given a right R-module M, the set of endomorphisms of M (i.e., right R-module homomorphisms from M to itself) forms a ring, $\operatorname{End} M_R$, with sums and products respectively defined by

$$(f+g)(m) = f(m) + g(m) \quad \text{and} \quad (fg)(m) = f(g(m)),$$

for all $m \in M$ and all endomorphisms f and g of M. (You do not have to verify that $\operatorname{End} M_R$ is a ring.) Prove that $\operatorname{End} R_R$, the ring of endomorphisms of the right R-module R, is isomorphic as a ring to R itself.

I.3. Let n be a positive integer, and let $M_n(\mathbb{C})$ denote the set of $n \times n$ complex matrices, viewed as a $\mathbb{C}$-vector space under scalar multiplication. Given an arbitrary matrix X in $M_n(\mathbb{C})$, let $P(X)$ denote the $\mathbb{C}$-linear span of the non-negative powers of X (i.e., $I, X, X^2, \ldots$), where I is the $n \times n$ identity matrix. Let $d(X)$ denote the dimension of $P(X)$ as a $\mathbb{C}$-linear subspace of $M_n(\mathbb{C})$. Find the least upper bound for $d(X)$, in terms of n, and provide a proof for your answer.

I.4. Is the subring $\mathbb{Q}[x, \sqrt{2}]$ of $\mathbb{R}[x]$ a principal ideal domain? If yes provide a proof and if no provide a counterexample (i.e., an ideal that is not principle).

Part II. Do two of these problems.

II.1. Let p and q be primes, and let G be a group of order p^2q. Prove that G is not simple. (You may not cite the Burnside p^aq^b Theorem.)

II.2. A ring is *left noetherian* when all of its left ideals are finitely generated. Consider the ring

$$R := \begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix} = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; \begin{array}{r} a, c \in \mathbb{Z} \\ b \in \mathbb{Q} \end{array} \right\}.$$

(You do not need to provide a proof that R is a ring.) Show that R is not noetherian.

II.3. Prove that the extension $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ is not Galois. Find the Galois closure E of the extension $\mathbb{Q}(\sqrt[3]{5})/\mathbb{Q}$ and find (at least one) primitive element of the extension $E/\mathbb{Q}$. Justify all the claims you are making.