Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group of order p^n (p is prime). Prove that G has a normal subgroup of order p^m for all $0 \leq m \leq n$.

I.2 Recall that a commutative ring R with identity $(1 \neq 0)$ is called local if it has exactly one maximal ideal.

(a) Prove that a commutative ring R with identity is *local* if and only if all non-units of R form an ideal of R; this is exactly the unique maximal ideal of R.

(b) Prove that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to a direct product of local rings for every $0 \neq n \in \mathbb{Z}$.

I.3 Let F be a field of characteristic 0 and let $V = \bigoplus_{i=0}^{n} Fx^i$ be the F-vector space of all polynomials of degree at most n. Let D be the endomorphism of V that is given by formal differentiation, $\frac{d}{dx}$.

(a) Find the Jordan canonical form of D.

(b) Determine all D-invariant subspaces of V.

I.4 Let F be a field and n a positive integer. Consider the polynomial ring $F[x_1, \ldots, x_n]$ and the ring R of all functions $F^n \to F$, with pointwise addition and multiplication of functions. Show that the evaluation map

$$\phi\colon F[x_1, \ldots, x_n] \to R\,, \qquad \phi(f)(\lambda_1, \ldots, \lambda_n) = f(\lambda_1, \ldots, \lambda_n)$$

is injective if and only if F is infinite.

Part II. Do two of these problems.

II.1 Let $\mathbb{F}_p$ be the finite field with p elements. A *maximal flag* in the $\mathbb{F}_p$-vector space $V = \mathbb{F}_p^n$ is a sequence of subspaces,

$$V = V_n \supset V_{n-1} \supset \cdots \supset V_2 \supset V_1 \supset V_0 = \{\mathbf{0}\},$$

where $\dim V_k = k$. Let U be the subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ which consist of elements g satisfying

- $g(V_k) = V_k$, and
- g induces the identity map on V_k/V_{k-1}

for all $n \geq k \geq 1$. Prove:

(a) U is a Sylow p-subgroup for every maximal flag.

(b) Every Sylow p-subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ is of this form.

(c) The number of Sylow p-subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$ is given by

$$n_p\big(\mathrm{GL}_n(\mathbb{F}_p)\big) = (1+p)(1+p+p^2)\dots(1+p+p^2+\cdots+p^{n-1}).$$

II.2 Let V be a finite-dimensional vector space over the algebraically closed field F. Recall that an endomorphism $\phi \in \mathrm{End}_F(V)$ is called *diagonalizable* if V has a basis consisting of eigenvectors for ϕ. Prove:

(a) ϕ is diagonalizable if and only if the minimal polynomial $m_\phi(t) \in F[t]$ is separable.

(b) If $W \subseteq V$ is a subspace such that $\phi(W) \subseteq W$, then the restriction $\phi\big|_W \in \mathrm{End}_F(W)$ is diagonalizable.

II.3 Let $\zeta = e^{2\pi i/7} \in \mathbb{C}$. Determine the degree of the following elements over $\mathbb{Q}$.

(a) $\zeta + \zeta^5$,

(b) $\zeta^3 + \zeta^5$,

(c) $\zeta^3 + \zeta^5 + \zeta^6$.