Comprehensive Examination in Algebra
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Part I. Solve three of the following problems.

I.1 Which of the following polynomials are irreducible? (Justify your answer.)

(a) $x^4 + 6x^3 - 4x^2 + 16x + 6 \in \mathbb{Z}[x]$

(b) $x^4 + 6x^3 - 4x^2 + 16x + 6 \in \mathbb{F}_5[x]$ (where $\mathbb{F}_5$ is the field of five elements)

(c) $x^3 - 3x^2 - 5x - 3 \in \mathbb{Z}[x]$

I.2 Let E be a field and $f(x), g(x) \in E[x]$ be irreducible *quadratic* polynomials. Set $K = E[x]/(f(x))$ and $L = E[x]/(g(x))$. Prove that

$$M = E[x, y]/(f(x), g(y))$$

is a field if and only if $K \not\cong L$.

I.3 Let $\alpha = \sqrt{3}$ and set

$$R = \mathbb{Z}[\alpha] = \{a + b\alpha \ : \ a, b \in \mathbb{Z}\}.$$

Consider the principal ideal $\mathcal{P} = (5)$ of R generated by 5.

(a) Prove that the quotient ring $R/\mathcal{P}$ is a field with twenty-five elements.

(b) Prove that $\mathcal{Q} = (11)$ is not a prime ideal of R.

I.4 Let G be a finite group acting transitively on a set X.

(a) Suppose that $|G| = 65$ and there is an element $g \in G$ of order 5 such that $g(x_0) = x_0$ for *some* $x_0 \in X$ (i.e., g has a fixed point on X). Show that $g(x) = x$ for *all* $x \in X$.

(b) Show that part (a) is false if $|G| = 60$: There is an action of the alternating group A_5 on a set X so that every 5-cycle in A_5 has a fixed point in X but no 5-cycle fixes every $x \in X$. (*Hint*: You can find such an X with $|X| = 6$.)

Part II. Solve two of the following problems.

II.1 Let p be a prime number, $\mathbb{F}_p$ be the field of p elements, and $G = \mathrm{GL}_2(\mathbb{F}_p)$ be the group of invertible 2×2 matrices with entries in $\mathbb{F}_p$.

(a) Prove that the group U_p of upper-triangular matrices with ones on the diagonal is a Sylow p-subgroup of G.

(b) Show that the normalizer of U_p in G is the group B_p of all upper-triangular matrices.

(c) Conclude that G has exactly $p+1$ Sylow p-subgroups.

II.2 Let E be the splitting field over $\mathbb{Q}$ of the polynomial $p(x) = x^7 - 3$.

(a) Compute the Galois group $\mathrm{Gal}(E/\mathbb{Q})$, either with a finite presentation or the abstract group structure, with a concrete description of the action on roots of $p(x)$.

(b) Find a primitive generator for E over $\mathbb{Q}$.

(c) Describe all the subfields of E that are Galois over $\mathbb{Q}$ as the subfield fixed by some subgroup of $\mathrm{Gal}(E/\mathbb{Q})$. You do not need to compute primitive generators for each field.

II.3 Let $\Phi_{12}(x) = x^4 - x^2 + 1 \in \mathbb{Q}[x]$.

(a) Prove that $p(x)$ is irreducible over $\mathbb{Q}$. (*Hint*: What are its roots in $\mathbb{C}$?)

(b) Give an explicit example of a 4×4 matrix $A \in \mathrm{M}_4(\mathbb{Q})$ with characteristic polynomial $p(x)$. *You must prove that your matrix has this characteristic polynomial.*

(c) Prove that $p(x)$ is also the minimal polynomial of A.

(d) Prove that your matrix A has order twelve, that is, $A^{12} = \mathrm{Id}$, where Id is the 4×4 identity matrix.