Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 (a) Prove that none of the following five groups of order 8 are isomorphic:

$$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3},\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/8\mathbb{Z},\ D_4 = \langle r, s \mid r^4, s^2, rsrs \rangle,\ Q_8 = \langle a, b \mid a^4, a^2b^2, b^{-1}aba \rangle.$$

(b) Prove that if G is a group of order 8, then G must be isomorphic to one of the groups listed in part (a).

I.2 Let R be a commutative ring with $1 \neq 0$ and such that for every $x \in R$, there is some natural number $n > 1$ such that $x^n = x$. Show that every prime ideal of R is maximal.

I.3 Let V be a finite-dimensional vector space over a field K and let $T \in \operatorname{End}_K(V)$ be a linear operator. Consider a family of subspaces $W_i \subseteq V$ $(i \in I)$ such that $T(W_i) \subseteq W_i$ and put $T_i = T\big|_{W_i} \in \operatorname{End}_K(W_i)$. With $m, m_i \in K[x]$ denoting the minimal polynomials of T and T_i, show:

(a) m_i divides m.

(b) If $V = \sum_{i \in I} W_i$, then m is the least common multiple of the m_i.

I.4 Let R be a commutative ring with identity and $f = \sum_{i=0}^{\infty} a_i x^i \in R[\![x]\!]$. Show that a_0 is a unit in R if and only if f is a unit in $R[\![x]\!]$.

Part II. Do two of these problems.

II.1 Prove that no group of order $105 = 3 \cdot 5 \cdot 7$ is simple.

II.2 Find the rank, elementary divisors, and invariant factors of the finitely generated $\mathbb{Z}$-module $\mathbb{Z}^3/\langle(2,4,8),(6,3,9)\rangle$.

II.3 Let F_1 and F_2 be Galois extensions of a field K, both contained in some fixed algebraic closure of K, and let $F = F_1F_2$ be the composite field. Show:

(a) F/K is Galois.

(b) The Galois group $\mathrm{Gal}(F/K)$ is abelian if and only if both $\mathrm{Gal}(F_i/K)$ are abelian.