Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group and let $A \trianglelefteq G$ be a normal subgroup. Assume that G/A is finite and that all nonidentity elements of A have infinite order. Also assume that A is *self centralizing*; that is,

$$A = C_G(A) = \{g \in G \mid ga = ag \text{ for all } a \in A\}.$$

Show:

(a) A is a maximal abelian subgroup of G.

(b) A is exactly the set of elements of G whose conjugacy class is finite.

I.2 Let F be a field and let $F^\times$ be the group of non-zero elements of F with respect to multiplication. Prove that every finite subgroup $G \leq F^\times$ is cyclic.

I.3 Let k be a field, and let $R = k[t]$. Let F be the quotient field of R. Prove that there exist finite field extensions of F of arbitrarily high degree.

I.4 Let R be a non-zero commutative ring (with 1) and let n, m be distinct positive integers. Prove that the free R-module R^n is not isomorphic to the free R-module R^m.

Part II. Do two of these problems.

II.1 Let $\mathbb{F}_3$ denote the field of order 3, with elements denoted 0, 1, and 2. Set $G = SL_2(\mathbb{F}_3)$, the group of 2×2 matrices with entries in $\mathbb{F}_3$ and determinant equal to 1. Let Z denote the center of G.

(a) Determine the order of G and the order of Z.

(b) Describe the group G/Z, by explicitly identifying it as a permutation group (i.e., by explicitly identifying it as a subgroup of S_n for some n).

Make sure that your answers are justified.

II.2 Let V be a finite-dimensional vector space over a field K with $\operatorname{ch} K = p > 0$ and let $L = \operatorname{End}_K(V)$ denote the ring of K-linear endomorphisms of V, with multiplication given by composition: $\phi\psi = \phi \circ \psi$ for $\phi, \psi \in L$. Let $\operatorname{ad}\phi \in \operatorname{End}_K(L)$ be defined by $(\operatorname{ad}\phi)(\psi) = \phi\psi - \psi\phi$. Prove the following equalities, for any $\phi \in L$:

(a) $\operatorname{trace}(\phi^p) = (\operatorname{trace}\phi)^p$.

(b) $\operatorname{ad}(\phi^p) = (\operatorname{ad}\phi)^p$.

II.3 Let E and L be finite field extensions of F contained in some common field. Let us assume that E is Galois over F. Prove:

(a) EL is Galois over L and E is Galois over $E \cap L$.

(b) The restriction map $\operatorname{Gal}(EL/L) \to \operatorname{Gal}(E/E\cap L)$, $g \mapsto g\big|_E$, is an isomorphism of groups.