Comprehensive Examination in Algebra
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August 2018

In these problems, groups need not be finite and, unless explicitly stated otherwise, rings need not be commutative. All rings and ring homomorphism are understood to be unital: each ring R has an identity element, 1_R, which is contained in all subrings of R, and $f(1_R) = 1_S$ holds for every ring homomorphism $f\colon R \to S$.

Part I. Do three of these problems.

I.1 Let G be a group, let N be a finite normal subgroup of G, and let H be a subgroup of G having finite index in G. Prove that if $|N|$ and $|G : H|$ are relatively prime, then $N \subseteq H$.

I.2 Let K be a field and let $T : K[x] \to K[x]$ be an automorphism of the polynomial ring $K[x]$ having the following property (*): $T(\lambda) = \lambda$ for all $\lambda \in K$.

- **a)** Prove that there exist $\alpha, \beta \in K$, $\alpha \neq 0$ such that $T(x) = \alpha x + \beta$.
- **b)** Deduce that the automorphisms of $K[x]$ satisfying (*) form a group under composition, which is isomorphic to the semi-direct product $K^\times \ltimes K$. Here $K^\times = (K \setminus \{0\}, \cdot)$ and $K = (K, +)$ are the multiplicative and the additive group of K, respectively, and $K^\times$ acts on K by multiplication.

I.3 Let R be a commutative integral domain. A left R-module M is said to be *divisible* if $rM = \{rm \mid m \in M\} = M$ for all nonzero $r \in R$.

- **a)** Prove that there exist divisible R-modules.
- **b)** Suppose that R is a PID that is not a field. Prove that no nonzero finitely generated left R-module is divisible.

I.4 Let R be a ring containing a field K as a central subring (i.e., $r\lambda = \lambda r$ for all $r \in R, \lambda \in K$). Let $a \in R$ be algebraic over K; that is, there exists a nonzero polynomial $p(x)$ in $K[x]$ such that $p(a) = 0$. Further suppose that $p(x)$ has been chosen to be of least degree.

- **a)** Show by example that $p(x)$ need not be an irreducible polynomial.
- **b)** Show that $K[a]$, the smallest subring of R containing a and K, is a field if and only if $p(x)$ is irreducible.

Part II. Do two of these problems.

II.1 For any group G, define $\Phi(G)$ to be the intersection of all maximal subgroups of G. Prove:

a) $\Phi(G)$ is a characteristic subgroup of G. ($\Phi(G)$ is called the *Frattini subgroup* of G.)

b) If G is nilpotent, then all maximal subgroups of G are normal and have prime index. Conclude that the derived subgroup $[G, G]$ is contained in $\Phi(G)$.

c) If G is finitely generated, then every proper subgroup $H \subsetneqq G$ is contained in a maximal subgroup. Conclude that if $G = H\Phi(G)$, then $G = H$.

II.2 A ring homomorphism $f\colon R \to S$ is called *centralizing* if the ring S is generated by $f(R)$ together with $C_S(f(R)) = \{s \in S \mid sf(r) = f(r)s \text{ for all } r \in R\}$. (For example, if f is surjective or S is commutative, then f is evidently centralizing.) Prove:

a) Composites of centralizing ring homomorphisms are centralizing.

b) If f is centralizing, then $f(I)S$ is an ideal (twosided) for every ideal I of R.

c) If f is centralizing, then the preimage $f^{-1}(P) = \{r \in R \mid f(r) \in P\}$ of every prime ideal P of S is a prime ideal of R.

II.3 Let E be the splitting field of $x^3 - 2$ over $\mathbb{Q}$.

a) Prove that $[E : \mathbb{Q}] = 6$ and the Galois group $G := \mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to S_3.

b) Find all intermediate fields of the extension $E/\mathbb{Q}$ corresponding to all four proper nontrivial subgroups of S_3.