Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let n be an odd integer ≥ 3 and G_n be the group with the following presentation

$$\langle r, s \mid r^n,\ s^2,\ srs^{-1}r \rangle.$$

Prove that the formulas

$$\varphi(r) := (1, 2, \ldots, n), \qquad \varphi(s) := (2, n)(3, n-1) \ldots \big((n+1)/2, (n+3)/2\big)$$

define a group homomorphism $\varphi : G_n \to S_n$, the symmetric group of degree n. Use this homomorphism to prove that G_n has order $2n$ and that φ is injective.

I.2 Consider the polynomial $p(x) := x^4 + x^3 + x^2 + x + 1 \in \mathbb{Q}[x]$.

a) Prove that $p(x)$ is irreducible over $\mathbb{Q}$.

b) Find a 4×4 matrix (with rational entries) whose characteristic polynomial is $p(x)$. *Please, do not forget to show that the characteristic polynomial of your matrix is indeed* $p(x)$.

I.3 Let F be a subring of an integral domain R. Prove that, if F is a field and R is a finite dimensional vector space over F, then R is also a field.

I.4 Let F be a field and let p be an integer > 1. A polynomial of the form $\sum_{i=0}^{n} f_i t^{p^i} \in F[t]$ is called a *p-polynomial*. Show that every $0 \neq f(t) \in F[t]$ is a factor of some nonzero p-polynomial.

Part II. Do two of these problems.

II.1 Let G be a finitely generated group and let n be a positive integer. Prove:

a) There are at most finitely many subgroups $H \leq G$ such that $|G : H| = n$.

b) For any $H \leq G$ with $|G : H| < \infty$, there is a characteristic subgroup of $C \leq G$ such that $C \leq H$ and $|G : C| < \infty$.

Recall that a subgroup $C \leq G$ is called characteristic if $\varphi(C) = C$ for all $\varphi \in \mathrm{Aut}(G)$. **Hint:** *Since G is finitely generated, there are only finitely many group homomorphisms from G to the symmetric group S_n.*

II.2 Let R be a left noetherian domain (not necessarily commutative). Show that any two $0 \neq x, y \in R$ have a nonzero common left multiple: $Rx \cap Ry \neq 0$. **Hint:** *Consider the chain $L_0 \subseteq L_1 \subseteq \ldots$ with $L_n = \sum_{i=0}^{n} Rxy^i$.*

II.3 Recall that, for every prime p, the Galois group of $\mathbb{Q}(e^{\frac{2\pi i}{p}})/\mathbb{Q}$ is isomorphic to the group of units of the ring $\mathbb{Z}/p\mathbb{Z}$ and the latter group is cyclic of order $p - 1$. Let $p = 11$ and put $\zeta := e^{\frac{2\pi i}{11}}$.

a) Find a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$.

b) Find primitive elements of the intermediate fields E_1, E_2 of $\mathbb{Q}(\zeta)/\mathbb{Q}$ corresponding to the two proper non-trivial subgroups of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong \mathbb{Z}/10\mathbb{Z}$.