Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1. Let G be a finite group. Say that a normal subgroup I of G is *cocyclic* provided G/I is a cyclic group, and let N denote the intersection of all of the cocyclic subroups of G. (i) Prove that G/N is isomorphic to a subgroup of a direct product of cyclic groups. (ii) Prove that G is abelian if and only if N is the trivial subgroup.

I.2. Let $\mathbb{Q}$ be the additive abelian group of rational numbers, and let H be a finitely generated subgroup of $\mathbb{Q}$. Prove that $\mathbb{Q}/H$ is not finitely generated.

I.3. Let n be a positive integer, let $M_n(\mathbb{C})$ denote the set of $n\times n$ complex matrices, and let I denote the $n\times n$ identity matrix. Now let ℓ be a positive integer less than n, and set

$$S = \{\ X \in M_n(\mathbb{C})\ :\ X^\ell = I\ \}.$$

Prove that S is a union of finitely many conjugacy classes in $M_n(\mathbb{C})$.

I.4. Let K denote the subfield $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ of $\mathbb{R}$. (i) Prove that K is a Galois extension of $\mathbb{Q}$. (ii) Find a primitive element μ for the extension $K/\mathbb{Q}$, and provide a proof that $K = \mathbb{Q}(\mu)$.

Part II. Do two of these problems.

II.1. Let G be a finite group with the following property: If H is a proper (i.e., strictly smaller) subgroup of G, then H is also properly contained in its normalizer $N_G(H)$.

(i) Prove that every Sylow subgroup of G is normal in G. (Hint: First verify that if P is a Sylow p-subgroup of G, then P is the unique Sylow p-subgroup of $N_G(P)$.)

(ii) Prove that G is isomorphic to the direct product of its Sylow subgroups.

II.2. Let R be a (not necessarily commutative) ring with identity. Suppose that R contains exactly one maximal left ideal M.

(i) Prove that there is exactly one isomorphism class of simple left R-modules.

(ii) Prove that M is a two sided ideal of R. (That is, prove that $Mr \subseteq M$ for all $r \in R$.)

(iii) Prove that there is exactly one isomorphism class of simple right R-modules.

II.3. Let K be the splitting field over $\mathbb{Q}$ of $x^6 + 3$. (i) Prove that $[K : \mathbb{Q}] = 6$. (ii) Prove that the Galois group of K over $\mathbb{Q}$ is not isomorphic to $\mathbb{Z}/6\mathbb{Z}$. (iii) Deduce that the Galois group of K over $\mathbb{Q}$ is isomorphic to S_3.