Comprehensive Examination in Algebra
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Part I. Do three of these problems:

I.1 Consider the ring $\mathbb{Z}[x]$. Give an example of a non-zero prime ideal in $\mathbb{Z}[x]$ that is not maximal. Now let R be an arbitrary principal ideal domain. Prove that every non-zero prime ideal in R is also maximal.

I.2 Let p be prime and let A be a $(p-1) \times (p-1)$ matrix with rational entries such that $A \neq I_{p-1}$, the $(p-1) \times (p-1)$ identity matrix, but

$$A^p = I_{p-1} \, .$$

Find the rational canonical form of A over $\mathbb{Q}$ and the Jordan canonical form of A over $\mathbb{C}$.

I.3 Let R be an integral domain. Prove that the polynomial ring $R[x]$ is also an integral domain. Prove that the units of $R[x]$ are precisely the units of R. Give an example of a commutative ring R (with $1 \neq 0$) and a pair of polynomials $p(x), q(x) \in R[x]$ of positive degrees for which

$$p(x) \cdot q(x) = 1 \, .$$

I.4 Find all isomorphism classes of groups G of order 12 which satisfy this property: *the Sylow* 3*-subgroup* H *of* G *is normal in* G *.*

Part II. Do two of these problems:

II.1 Let R be a principal ideal domain. Prove directly (i.e., without invoking Hilbert's Basis Theorem) that every ideal in the polynomial ring $R[x]$ is finitely generated.

Hint: Let I be a non-zero ideal of $R[x]$. Prove that the union

$$L = \{0\} \cup \{ \text{ leading coefficients of non-zero polynomials in } I \}$$

is an ideal of R. Since R is a principal ideal domain, $L = (r)$ for some $r \in R$ and there is a polynomial $f(x) \in I$ whose leading coefficient is r. Next, consider polynomials in I of degrees $< \deg f(x)$.

II.2 Let $K \supset F$ be a finite separable field extension. Let us denote by $\mathrm{Emb}(K/F)$ the set of all ring homomorphisms

$$\psi : K \to \overline{F},$$

such that $\psi(a) = a$ for every $a \in F$, where $\overline{F}$ is a fixed algebraic closure of F. Prove that the number of elements in $\mathrm{Emb}(K/F)$ is precisely $[K : F]$.

II.3 Let G be a group having a series of subgroups $1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$ such that all G_i are normal in G and all factors G_i/G_{i-1} are cyclic. (Such a group G is called *supersolvable*.) Show that G contains a nilpotent normal subgroup of finite index. Give an example of a non-nilpotent supersolvable group.