Comprehensive Examination in Algebra
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Part I. Do three of these problems.

I.1 Let G be a group (not necessarily finite). For each $g \in G$, recall that the elements hgh^{-1}, for $h \in G$, are the *conjugates* of g. Put

$$\mathcal{FC}(G) := \{g \in G \mid g \text{ has only finitely many conjugates}\}.$$

(a) Prove that $\mathcal{FC}(G)$ is a characteristic subgroup of G. (Recall that a subgroup $H \leq G$ is *characteristic* if H is mapped to itself under all automorphisms of G.)

(b) Show that all finite normal subgroups of G are contained in $\mathcal{FC}(G)$.

(c) Consider the infinite dihedral group $D_\infty = \langle x, y \mid y^2 = 1, yx = x^{-1}y \rangle$. Prove that $y \notin \mathcal{FC}(D_\infty)$. Conclude that an arbitrary finite subgroup of a group G need not be contained in $\mathcal{FC}(G)$.

I.2 Let V be a vector space over a field F and let φ be an F-linear operator on V. Prove there exists an F-linear operator ψ on V satisfying $\varphi\psi\varphi = \varphi$, by completing the following steps:

(a) If C is a complement of $\mathrm{Ker}(\varphi)$ in V then we have an isomorphism $\varphi\big|_C \colon C \xrightarrow{\sim} \mathrm{Im}(\varphi)$.

(b) If an F-linear operator ψ on V satisfies $\psi\big|_{\mathrm{Im}(\varphi)} = (\varphi\big|_C)^{-1}$ then $\varphi\psi\varphi = \varphi$.

I.3 Let R be a commutative ring (with identity $1 \neq 0$). Assume further that $P_1, \ldots, P_n$ are finitely many prime ideals of R such that the intersection $N = P_1 \cap \cdots \cap P_n$ is nilpotent, that is, $N^t = 0$ for some $t \geq 0$. Prove that every prime ideal of R must contain one of the $P_1, \ldots, P_n$.

I.4 Let F be a field of characteristic $p > 0$ and let $f(X) \in F[X]$ be an irreducible polynomial. Show that $f(X)$ is separable (i.e., $f(X)$ has repeated roots) if and only if $f(X) \notin F[X^p]$.

Part II. Do two of these problems.

II.1 Let A_n denote the alternating group of degree $n \geq 5$.

(a) Show that A_n acts transitively on $\{1, \ldots, n\}$. Conclude that A_n has a subgroup of index n.

(b) Show that A_n has no proper subgroup of index $< n$.

II.2 Let R be a ring (with $1 \neq 0$ but not necessarily commutative). Assume that $x, y \in R$ are given such that $xy = 1$. Put $e = yx \in R$.

(a) Show that $e = e^2$ and conclude that $R = eR \oplus (1 - e)R$.

(b) Show that the map $er \mapsto xer$ $(r \in R)$ yields an isomorphism of right R-modules, $eR \cong R$.

(c) Conclude that if R contains no infinite direct sums of nonzero right ideals then $e = 1$.

II.3 Let F/K be a Galois extension of fields having characteristic $\neq 2$. Assume that $\mathrm{Gal}(F/K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Prove that $F = K(\sqrt{a}, \sqrt{b})$ for some $a, b \in K$.