Another Constraint on the Perfect Cuboid

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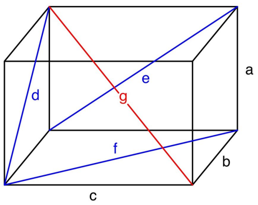
What is the Perfect Cuboid?

The Perfect Cuboid



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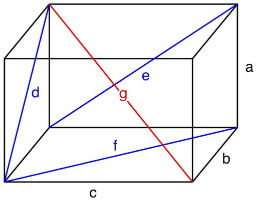

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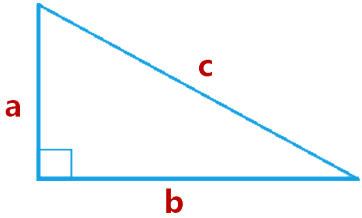
where $a, b, c, d, e, f, g \in \mathbb{N}$

A quick history lesson.

The Pythagorean Triangle.



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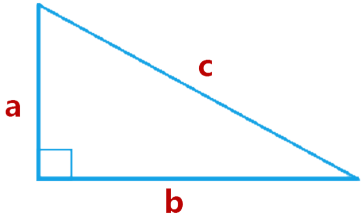

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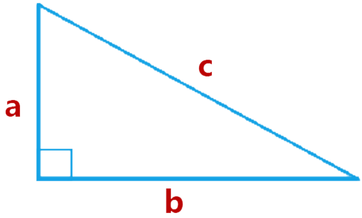

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  - $(a, b, c) = (m^2 - n^2,\ 2mn,\ m^2 + n^2)$

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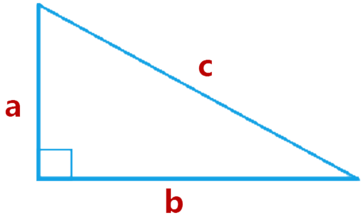

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  - $(a, b, c) = (m^2 - n^2,\ 2mn,\ m^2 + n^2)$ where $m, n \in \mathbb{Z}$
  - $m > n,\ m \not\equiv n \mod 2,\ gcd(m, n) = 1$

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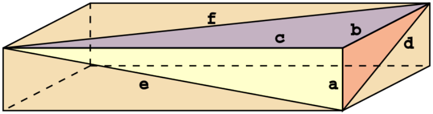
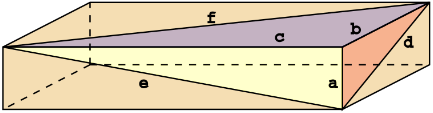
The Euler Brick





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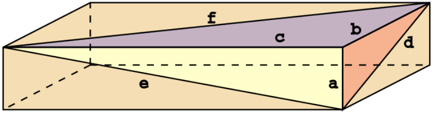

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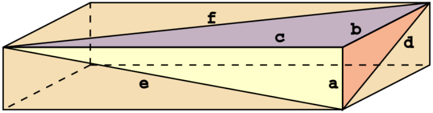

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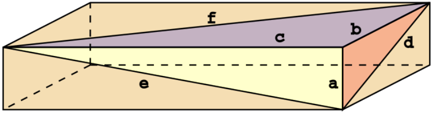

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- $(a,b,c)=(240,252,275)$ and $(d,e,f)=(348,365,373)$

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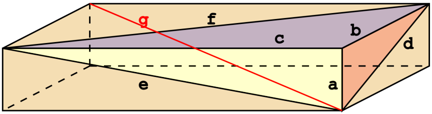

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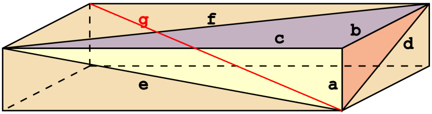

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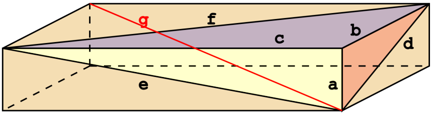

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- With the new constraint: $a^2 + b^2 + c^2 = g^2$

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- For $p \in \{5, 7, 11, 19\}$, one edge must be divisible by p.
- One edge or space diagonal must be divisible by 13.
- For $p \in \{17, 29, 37\}$, one edge, face diagonal or space diagonal must be divisible by p.

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- Can we add another prime?
- Can we raise the power of one of these primes?

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Goal: Raise one of these known prime divisors, $7 \leq p_i \leq 37$.
We will look to see if there exists an n, such that $7^n | P$.

The Reduction Mod n

$$A = \begin{cases} a^2 + b^2 = d^2 \\ a^2 + c^2 = e^2 \\ b^2 + c^2 = f^2 \\ a^2 + b^2 + c^2 = g^2 \end{cases}$$

The Main Idea

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$$A = \begin{cases} a^2 + b^2 = d^2 \\ a^2 + c^2 = e^2 \\ b^2 + c^2 = f^2 \\ a^2 + b^2 + c^2 = g^2 \end{cases} \implies A \equiv \begin{Bmatrix} a^2 + b^2 = d^2 \\ a^2 + c^2 = e^2 \\ b^2 + c^2 = f^2 \\ a^2 + b^2 + c^2 = g^2 \end{Bmatrix} \mod n$$

The Reduction Mod n

$$\implies A = \begin{cases} a^2 + b^2 \equiv d^2 \mod n \\ a^2 + c^2 \equiv e^2 \mod n \\ b^2 + c^2 \equiv f^2 \mod n \\ a^2 + b^2 + c^2 \equiv g^2 \mod n \end{cases}$$

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The Reduction Mod n

And if we know the set of Quadratic Residues mod $n := QR$

$$\implies A = \begin{cases} a^2 + b^2 \equiv d^2 \mod n \\ a^2 + c^2 \equiv e^2 \mod n \\ b^2 + c^2 \equiv f^2 \mod n \\ a^2 + b^2 + c^2 \equiv g^2 \mod n \end{cases} \implies \begin{cases} a + b \in QR \\ a + c \in QR \\ b + c \in QR \\ a + b + c \in QR \end{cases}$$

where $a, b, c \in QR$ themselves.

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2^2 &\equiv x \mod p \\
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4^2 &\equiv x \mod p \\
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&\vdots \\
(x^2 &\equiv x \mod p)
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So given $p \in \mathbb{N}$, what is x^2 for $x \in \{0, 1, 2, 3, \ldots\}$? For $p = 7$,

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3^2 &\equiv x \mod p & \implies 9 &\equiv 2 \mod 7 \\
4^2 &\equiv x \mod p & \implies 16 &\equiv 2 \mod 7 \\
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So we get $QR \mod 7 = \{0, 1, 2, 4\}$

Quadratic Residues

Psuedocode for $QR \mod 7 = \{0,1,2,4\}$

```
Enter p=7
for n=1: floor(p/2)
        QR(n)= rem(n^2,p)
end

QR=[0,B]        %adding 0, and sorting it
QR=sort(QR)

print(QR)={0,1,2,4}
```

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For $QR \mod 7 = \{0, 1, 2, 4\}$. The only C vectors that work are,
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But notice that all combinations have a 0, so we can conclude that at least one square is divisible by 7, and therefore at least one edge is divisible by 7.

Combinations

Psuedocode: $C = QR \times QR \times QR = (QR)^3$

```
A=allcomb(QR,QR,QR)
        %First we make all combinations of C

H= sum("all columns" of A)
        %we are checking a+b+c in QR?
        %(for each row/vector/combination)
A=
        0,0,0
        0,0,1
        0,0,2
        0,0,4
        0,1,0
        ...
H= 0,1,2,4,1,2,3,5,...
```

Passes

Psuedocode: Keeping what we want, Deleting the rest

```
Pass123=[vector (we don't know yet)]

for i = 1 to length(H)
        for k = 1 to length(QR)
                if H mod p \in QR
                Pass123 = [Pass123  i]
                %this counts the index of each H (sum)
        end
end

A_123=A(Pass123,:)
        % we choose the Pass123(i) rows of A, and keep them.
        %the rest are just "deleted"
```

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Note, if final array has combinations, then there exist combinations such that 0 isn't a part of it, so we cannot conclude that p is a divisor. (it can still be shown that p is a divisor in other ways, just not with modular arithmetic).

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As I was checking and running larger primes past 100, my professor had the great idea of checking 7^2.

Lo and behold it works, an empty final array $\implies$ all combinations had 0 (or $\equiv 0 \mod p$) $\implies$ the edge is divisible by 7^2.

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  - Result: $\left(\frac{2,3}{p}\right) = 1 \iff p \equiv \pm 1 \mod 24$

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    - $p \equiv 11 \mod 12 \wedge p \equiv 7 \mod 8$
  - Result: $\left(\frac{2,3}{p}\right) = 1 \iff p \equiv \pm 1 \mod 24$
- The reason why this is less impressive, is that it just means if you were to check the divisors $p \equiv 1 \mod 24$, you would know that they could never be a divisor of the perfect cuboid. Not as "cool" as finding divisors.

References and Thanks

References

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