Ph.D. Comprehensive Examination
Partial Differential Equations
August 2012

Part I. Do three of these problems.

I.1 Use the method of characteristics to find a solution of

$$\begin{cases} \dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0 \\ u(0,y) = \sin(y). \end{cases}$$

I.2 Show that $u(t) = |t|^{1/2}$ $(t \in \mathbb{R})$ is a weak solution of

$$(t\frac{d}{dt} - \frac{1}{2})u = 0$$

in $\mathbb{R}$.

I.3 Define $K : \mathbb{R}^2 \to \mathbb{R}$ by $K(x,y) = H(x)H(y)$ where $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside function. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$. Show that if $f \in L^p(\Omega)$ $(1 \le p \le \infty)$ then

$$F(x,y) = \int_\Omega K(x-\xi, y-\eta) f(\xi,\eta)\, d\xi\, d\eta$$

belongs to $W^{1,p}(\Omega)$. Recall that $H(t) = 1$ if $t \ge 0$, $H(t) = 0$ otherwise.

I.4 Let $\lambda > 0$, define $E_\lambda : C(\mathbb{R}^2) \to \mathbb{C}$ by

$$E_\lambda(f) = \int_0^{2\pi} f(\lambda\cos\theta, \lambda\sin\theta)\, d\theta.$$

For $x \in \mathbb{R}^2$ let $w_x(\xi) = e^{ix\cdot\xi}$, so $w_x \in C(\mathbb{R}^2)$ for each x; here $i = \sqrt{-1}$. Define $u : \mathbb{R}^2 \to \mathbb{C}$ by

$$u(x) = E_\lambda(w_x).$$

a) Give a detailed argument as to why $u \in C^2$.

b) Let Δ be the standard Laplacian in $\mathbb{R}^2$. Show that $\Delta u = -\lambda^2 u$.

Part II. Do two of these problems.

II.1 Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with C^1 boundary, let $V \in C^2(\Omega)$ be *real-valued*, and suppose that $\psi \in C^2(\mathbb{R} \times \overline{\Omega})$ satisfies

$$\frac{1}{i}\frac{\partial \psi}{\partial t} = -\Delta\psi + V\psi \quad \text{in } \mathbb{R} \times \Omega, \quad \psi = 0 \text{ on } \mathbb{R} \times \partial\Omega.$$

Show that

$$\int_\Omega |\psi(t,x)|^2 \, dx,$$

a function of $t \in \mathbb{R}$, is constant. Here Δ is the standard Laplacian in the x variables and $i = \sqrt{-1}$. Hints: Differentiate in t, pay attention to real and imaginary parts, use the divergence theorem at some point; remember to justify each step.

II.2 Suppose u is a C^2 function on $\mathbb{R}^2 = \mathbb{R}_t \times \mathbb{R}_x$ such that

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0.$$

Show that if there is $r > 0$ such that $u(t,x) = 0$ for $|x| > r$ and $t \in \mathbb{R}$, then $u \equiv 0$.

II.3 Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and let $\Omega \subset \mathbb{R}$ be the interval (α, β). Further, let a, $b \in C(\overline{\Omega})$ be real-valued with a strictly positive, and define

$$Lu = -a\frac{d^2u}{dx^2} + b\frac{du}{dx} \quad \text{on } \Omega.$$

Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Show:

If $Lu \leq 0$ and $\max_{\overline{\Omega}} u$ is attained in Ω, then u is constant.

Hint: Suppose the maximum M is attained at a point $x_0 \in \Omega$ and $u \not\equiv M$. Consider the auxiliary function $z(x) = \exp(\lambda(x - x_0)) - 1$, and choose λ large enough so that $Lz < 0$ in Ω. Use $w(x) = u(x) + \varepsilon z(x)$, with $\varepsilon > 0$ sufficiently small, to obtain a contradiction.

Added Aug 30