PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Solve

$$-x\,u_x + u_y = (1-x^2)\,u$$
$$u(0,y) = 3\,e^y.$$

2. For $f \in L^1(\mathbb{R})$ recall that its Fourier transform $\mathcal{F} f(x) = \int_{\mathbb{R}} f(t)\,e^{-2\pi i\,x\,t}\,dt$.

   Let $a > 0$. Prove that $\mathcal{F}\left(e^{-a\pi|x|^2}\right)(\xi) = a^{-n/2}e^{-\pi|\xi|^2/a}$. Conclude that (* denotes convolution)

$$e^{-\pi|x|^2} * e^{-\pi|x|^2} = 2^{-n/2}e^{-\pi|x|^2/2}.$$

3. Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and let $c(x)$ be a continuous positive function in $\bar{\Omega}$. Consider the boundary value problem

$$\frac{1}{c(x)^2}\,u_{tt} = \Delta_x u \qquad \text{for } x \in \Omega,\ t > 0$$
$$u_t - \alpha(x)\,\frac{\partial u}{\partial \nu} = 0 \quad \text{for } x \in \partial\Omega \text{ and } t > 0,$$

   α is a continuous function in $\partial\Omega$. Let $E(t) = \dfrac{1}{2}\displaystyle\int_{\Omega}\left(\frac{1}{c(x)^2}\,u_t^2 + |\nabla_x u|^2\right)dx$.

   Prove that $\dfrac{dE}{dt} \geq 0$ if $\alpha(x) \geq 0$ for $x \in \partial\Omega$; and $\dfrac{dE}{dt} \leq 0$ if $\alpha(x) \leq 0$.

4. Let $b \in \mathbb{R}^n$, $c \in \mathbb{R}$, and let u be a solution to

$$u_t + b\cdot\nabla_x u + c\,u = \Delta_x u \quad \text{for } x \in \mathbb{R}^n \text{ and } t > 0$$
$$u(x,0) = f(x) \quad \text{for } x \in \mathbb{R}^n.$$

   Find constants $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $u(x,t) = e^{\alpha\cdot x+\beta t}\,v(x,t)$ with v satisfying the heat equation $v_t - \Delta_x v = 0$. Find $v(x,0)$.

Part II. (Do 2 problems)

1. Let $f, g \in W^{1,2}(\Omega)$. Prove that $fg \in W^{1,1}(\Omega)$ and $D(fg) = fDg + gDf$.

   HINT: from Meyers-Serrin theorem there exist $f_n, g_n \in C^\infty(\Omega) \cap W^{1,2}(\Omega)$ with $f_n \to f$ and $g_n \to g$ in $W^{1,2}(\Omega)$. Show that $f_n g_n \to fg$ in $L^1(\Omega)$ and $f_n Dg_n + g_n Df_n \to fDg + gDf$ in $L^1(\Omega)$. Since f_n, g_n are smooth $D(f_n g_n) = f_n Dg_n + g_n Df_n$, conclude the result.

2. Let $\Omega \subset \mathbb{R}^n$ with smooth boundary. Prove that the boundary value problem

$$\begin{aligned} \Delta u + \alpha(x)\, u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

   cannot have more than one smooth solution provided $\|\alpha\|_{L^\infty(\Omega)}$ is sufficiently small.

   HINT: Use Poincaré's inequality $\|u\|_2 \le C_1 \|\nabla u\|_2$ for all $u \in C_0^1(\Omega)$.

3. Let u be biharmonic in $\mathbb{R}^n$, i.e., $\Delta^2 u = \Delta(\Delta u) = 0$. Prove that u satisfies the following mean value property

$$⨍_{|x|=r} u(x)\, d\sigma(x) = u(0) + \frac{r^2}{2\,n}\, \Delta u(0)$$

   for all $r > 0$.

   HINT: Δu is harmonic, then use the solid mean value property for harmonic functions, the divergence theorem and integrate the resulting identity from 0 to r.

ANSWER: Δu is harmonic, so from the solid mean value property

$$\begin{aligned}
\Delta u(0) &= \frac{1}{|B_r(0)|} \int_{B_r(0)} \Delta u(x)\, dx \\
&= \frac{1}{|B_r(0)|} \int_{|x|=r} \frac{\partial u}{\partial \nu}\, d\sigma(x) \quad \text{from the divergence theorem} \\
&= \frac{1}{|B_r(0)|} \int_{|x|=r} \nabla u(x) \cdot \frac{x}{r}\, d\sigma(x) \\
&= \frac{1}{|B_r(0)|} \int_{|z|=1} \nabla u(r\,z) \cdot \frac{r\,z}{r}\, r^{n-1} d\sigma(z) \\
&= \frac{1}{|B_r(0)|}\, r^{n-1} \int_{|z|=1} \frac{d}{dr}(u(r\,z))\, d\sigma(z) \\
&= \frac{1}{|B_r(0)|}\, r^{n-1} \frac{d}{dr}\left(\int_{|z|=1} u(r\,z)\, d\sigma(z)\right) \\
&= \frac{1}{|B_r(0)|}\, r^{n-1} \frac{d}{dr}\left(\frac{1}{r^{n-1}} \int_{|x|=r} u(x)\, d\sigma(x)\right) \\
&= \frac{\omega_{n-1}\, r^{n-1}}{|B_r(0)|} \frac{d}{dr}\left(\frac{1}{\omega_{n-1}\, r^{n-1}} \int_{|x|=r} u(x)\, d\sigma(x)\right) \\
&= \frac{\omega_{n-1}\, r^{n-1}}{|B_r(0)|} \frac{d}{dr}\left(⨍_{|x|=r} u(x)\, d\sigma(x)\right) \\
&= \frac{n}{r} \frac{d}{dr}\left(⨍_{|x|=r} u(x)\, d\sigma(x)\right),
\end{aligned}$$

so we obtain

$$\frac{r}{n}\, \Delta u(0) = \frac{d}{dr}\left(⨍_{|x|=r} u(x)\, d\sigma(x)\right).$$

Integrating the last identity between 0 and r yields

$$\frac{r^2}{2\,n}\, \Delta u(0) = ⨍_{|x|=r} u(x)\, d\sigma(x) - u(0).$$