PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Solve the initial value problem

$$-x\,u_x + y\,u_y = x\,u^2, \qquad u(s,1) = e^{-s}.$$

2. Let $f \in L^1(\mathbb{R}^n)$. Prove that the Fourier transform $\hat{f}$ is uniformly continuous in $\mathbb{R}^n$.

3. Show that when $n = 2$, the function $u(x) = -\dfrac{1}{8\pi}|x|^2 \log|x|$ is a fundamental solution to the bi-harmonic operator $\Delta^2 = \Delta(\Delta)$. That is, show that

$$\varphi(0) = \int_{\mathbb{R}^2} u(x)\Delta^2\varphi(x)\,dx,$$

for all functions φ smooth with compact support. HINT: show that $\Delta u = -\dfrac{1}{2\pi}(1 + \log|x|)$.

4. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a constant vector, prove that $u(x,t) = \exp(\pm i(\alpha \cdot x + \omega\, c\, t))$ solves the wave equation $u_{tt} - c^2\Delta u = 0$ provided $|\alpha|^2 = \omega^2$. Here c and ω are real constants.

Part II. (Do 2 problems)

1. If $f \in W^{1,2}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open connected and $Df = 0$, then prove that f is constant in Ω (D denotes the gradient).

   HINT: let ϕ be smooth, nonnegative, with support on the unit ball and integral one. Take $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ and $f_\epsilon = f \star \phi_\epsilon$. Show first that f_ϵ is constant for each ϵ (constant depending on ϵ).

2. Let $u(x,t)$ be a solution to the heat equation $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, +\infty)$. Suppose that $\sup_{|x|<R} |u(x,t) - A(x)| \to 0$ as $t \to +\infty$ for some function $A(x)$. Prove that A is harmonic in $|x| < R$.

   HINT: prove that A is weakly harmonic in $|x| < R$, that is, $\int_{\mathbb{R}^n} A(x)\,\Delta\phi(x)\,dx = 0$ for all $\phi \in C_0^\infty(|x| < R)$. Using the equation and the divergence theorem show first that $\int_{t_1}^{t_2}\int_{\mathbb{R}^n} u(x,t)\,\Delta\phi(x)\,dxdt = \int_{\mathbb{R}^n}\phi(x)\,(u(x,t_2) - u(x,t_1))\,dx$. Next write $\int_{t_1}^{t_2}\int_{\mathbb{R}^n} A(x)\,\Delta\phi(x)\,dxdt = \int_{t_1}^{t_2}\int_{\mathbb{R}^n}(A(x) - u(x,t))\,\Delta\phi(x)\,dxdt + \int_{t_1}^{t_2}\int_{\mathbb{R}^n} u(x,t)\,\Delta\phi(x)\,dxdt$.

3. Suppose $u \in C^2(\bar{\Omega})$ is harmonic, with Ω a bounded and smooth connected domain; ν denotes the outer unit normal to $\partial\Omega$.

   (a) Prove that

   $$\int_{\partial\Omega} u\,\frac{\partial u}{\partial \nu}\,d\sigma(x) \geq 0,$$

   with strict inequality unless u is constant.
   HINT: apply the divergence theorem to the field uDu.

   (b) Deduce that the problem $\Delta u = f$ has at most one solution $u \in C^2(\bar{\Omega})$ satisfying $\dfrac{\partial u}{\partial \nu} + \alpha\,u = \beta$ on $\partial\Omega$, where $\alpha(x)$ and $\beta(x)$ are measurable functions in $\partial\Omega$, with $\alpha(x) > 0$ a.e.