PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Solve the Cauchy Problem

$$x\,\frac{\partial u}{\partial x} + (y+1)\,\frac{\partial u}{\partial y} = 3\,u$$
$$u(x,0) = f(x)$$

   for $y > -1$, where $f \in C^1(\mathbb{R})$.

2. The Fourier transform is defined by $\hat{f}(x) = \int_{\mathbb{R}^n} f(t)\, e^{-2\pi i\, x\cdot t}\, dt$. Suppose $f \in L^1(\mathbb{R}^n)$. Prove that

   (a) $\hat{f}$ is uniformly continuous in $\mathbb{R}^n$; and

   (b) $\lim_{|x|\to\infty} \hat{f}(x) = 0$.

3. Let $u(x) \geq 0$ and $u \in C^2(\bar{\Omega})$, with Ω a bounded smooth domain. If

$$\Delta u = u^2 \text{ on } \Omega \text{ and } u(x) = 0 \text{ on } \partial\Omega,$$

   then prove that $u \equiv 0$.
   Hint: Multiply the equation by u and integrate.

4. Consider the wave equation with damping

$$u_{tt} - u_{xx} + u_t = 0,\;\; x \in \mathbb{R}, t > 0,\;\; u(x,0) = f(x),\; u_t(x,0) = g(x), x \in \mathbb{R},$$

   with g having compact support. Let $E(t) = \displaystyle\int_{-\infty}^{\infty} \left(u_t^2 + u_x^2\right) dx$ be the energy of the solution at time t. Prove that $E(t)$ is a non increasing function.

Part II. (Do 2 problems)

1. If $\Omega = B(0,1)$ is the unit ball in $\mathbb{R}^n$ and $u(x) = \dfrac{1}{|x|^\alpha}$, show that $u \in W^{1,p}(\Omega)$ if and only if $\alpha < \dfrac{n}{p} - 1$.

2. Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$. Show that if $u \in C^2(B) \cap C(\overline{B})$, $u(x) = 0$ for $|x| = 1$, and $|\Delta u| \leq C$, where $C > 0$ is a constant, then

$$\frac{C}{2n}\left(|x|^2 - 1\right) \leq u(x) \leq -\frac{C}{2n}\left(|x|^2 - 1\right), \text{ for all } x \in B.$$

   Hint: Let $w(x) = \dfrac{C}{2n}(|x|^2 - 1)$. Show that $\Delta w = C$. Let $v = u - w$ and show that $\Delta v \leq 0$, i.e., v is super harmonic. By the minimum principle for super harmonic functions, $\min_{\bar{B}} v = \min_{\partial B} v$. Conclude the first inequality. To show the second inequality take $w(x) = -\dfrac{C}{2n}(|x|^2 - 1)$, $u - w$ is sub harmonic and proceed similarly.

3. Using the field $(x_1, \cdots, x_n, x_{n+1})$ and the divergence theorem, prove that the volume V of any bounded domain $D \subset \mathbb{R}^{n+1}$ for which the divergence theorem holds equals

$$V = \frac{1}{n+1} \int_{\partial D} |X - P| \cos(X - P, \nu)\, d\sigma(X),$$

   where P is any fixed point in $\mathbb{R}^{n+1}$ and $(X - P, \nu)$ denotes the angle between $X - P$, and ν is the outer unit normal to ∂D at the point X.

   Conclude that if C is a cone in $\mathbb{R}^{n+1}$ with sufficiently smooth base $\Omega \subset \mathbb{R}^n$ and height h, then the $n + 1$-dimensional volume of C equals

$$\frac{|\Omega|\, h}{n+1}$$

   where $|\Omega|$ is the n-dimensional volume of the set Ω.

   Hint: use the first part with P=vertex of the cone.