Ph.D. Comprehensive Examination (Sample II)
Partial Differential Equations

Part I. Do three of these problems.

I.1. Write a formula for the solution of the following initial-value problem

$$\begin{cases} u_t + a \cdot Du + bu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Here $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$ are constants and $g(x)$ is a continuous function.

I.2. Let $0 < R_1 < R_2$, $\Omega = \{x \in \mathbb{R}^n : R_1 < |x - x_0| < R_2\}$, and c_1, c_2 be real numbers. Find a harmonic function u on Ω such that $u = c_1$ on $|x - x_0| = R_1$ and $u = c_2$ on $|x - x_0| = R_2$.

I.3. Prove that the function $|x|^{-\alpha} \in W^{1,p}(B_1(0))$ if only if $\alpha < \frac{n-p}{p}$. Here $B_1(0)$ denotes the unit ball in $\mathbb{R}^n$ centered at the origin.

I.4. Let

$$f(x) = \begin{cases} e^{-ax} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

where $a > 0$. For $\xi \in \mathbb{R}$, compute

$$\int_{-\infty}^{\infty} e^{ix\xi} f(x)\, dx.$$

Part II. Do two of these problems.

II.1. Let $u \in C^2(\mathbb{R}^n)$ be a function of compact support. Prove that

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x - y)\, \Delta u(y)\, dy$$

where Γ is the fundamental solution of the Laplace operator.
Hint: Use a Green's formula.

II.2. Let $u \in C^2(\mathbb{R} \times [0, \infty))$ be a solution of the telegraph equation

$$u_{tt} = c^2 u_{xx} - hu$$

on $\mathbb{R} \times (0, \infty)$ where $c, h > 0$. Let $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$ and assume that $u(x, 0) = u_t(x, 0) = 0$ on the set $\{x : |x - x_0| \leq ct_0\}$. Prove that $u(x, t) = 0$ on the triangle $\{(x, t) : 0 \leq t < t_0,\ |x - x_0| \leq c(t_0 - t)\}$.

Hint: For $0 \leq t < t_0$, consider the energy

$$W(t) = \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} (u_t^2 + c^2 u_x^2 + hu^2)\, dx.$$

Observe that $W(0) = 0$ and $W(t) \geq 0$. Show that $W(t)$ is a decreasing function.

II.3. Let u be a harmonic function on a domain $\Omega \subseteq \mathbb{R}^n$, $u \in C(\overline{\Omega})$. Prove that

$$|Du(x)| \leq \frac{C_n}{\operatorname{dist}(x, \partial\Omega)} \max_{\overline{\Omega}} |u|$$

for all $x \in \Omega$, where C_n is a constant depending only on the dimension n.