PDEs Ph.D. Qualifying Exam
Temple University
August 19, 2019

Part I. (Do 3 problems)

1. Solve

$$\begin{aligned} u_x + x^2\, y\, u_y &= -u \\ u(0, y) &= y^2. \end{aligned}$$

2. Suppose $f \in L^1(\mathbb{R}) \cap C^2(\mathbb{R})$ with $f'' \in L^1(\mathbb{R})$. Prove that $x^2 \hat{f}(x) \to 0$ as $|x| \to \infty$.

3. Consider the biharmonic operator $\Delta^2 u = \Delta(\Delta u)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. Prove that Δ^2 is self-adjoint, that is,

$$\int_\Omega u\, \Delta^2 v\, dx = \int_\Omega v\, \Delta^2 u\, dx$$

for all $u, v \in C^4(\bar{\Omega})$ with $u = v = 0$ and $\Delta u = \Delta v = 0$ on $\partial\Omega$.

4. Let u be harmonic on a bounded domain Ω in $\mathbb{R}^n$, $u \in C(\bar{\Omega})$. Prove that

$$|Du(x)| \le \frac{C_n}{\text{dist}\,(x, \partial\Omega)} \max_{\Omega} |u|$$

for all $x \in \Omega$, where C_n is a constant depending only on the dimension n.

Part II. (Do 2 problems)

1. Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain and $u \in W^{1,2}(\Omega)$. Prove that $v(x) = F(u(x)) \in W^{1,2}(\Omega)$ and the weak derivatives $v_{x_i} = F'(u)u_{x_i}$ for $1 \le i \le n$.

2. Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $\Delta w = C_1$ in Ω and $w = 0$ on $\partial\Omega$ with C_1 a real constant.

   If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $|\Delta u| \le C_1$ in Ω and $u = 0$ on $\partial\Omega$, then prove that

$$w(x) \le u(x) \le -w(x), \text{ for all } x \in \overline{\Omega}.$$

3. Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. Let $u(x, t)$ be a smooth solution to

$$\begin{aligned} u_{tt} - \Delta u + u^3 &= 0 \text{ for } (x, t) \in \Omega \times [0, T] \\ u(x, t) &= 0 \text{ for } (x, t) \in \partial\Omega \times [0, T], \end{aligned}$$

and set $E(t) = \int_\Omega \left(u_t(x, t)^2 + |\nabla_x u(x, t)|^2\right) dx$.

   (a) Prove that $\dfrac{dE}{dt}(t) = -\dfrac{1}{2}\left(\int_\Omega u(x, t)^4\, dx\right)_t$ for $0 \le t \le T$.

   (b) Conclude that if $u(x, 0) = u_t(x, 0) = 0$ for $x \in \Omega$, then $u \equiv 0$.