PDEs Ph.D. Qualifying Exam
Temple University
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Part I. (Do 3 problems)

1. Solve

$$u_x + x^2\, y\, u_y = -u$$
$$u(0,y) = y^2.$$

2. For $f \in L^1(\mathbb{R})$ recall that its Fourier transform $\hat{f}(x) = \int_{\mathbb{R}} f(t)\, e^{-2\pi i x t}\, dt$. Consider

$$f(t) = \begin{cases} \dfrac{e^{2\pi i \alpha t}}{y} & \text{for } |t| \le y \\ 0 & \text{for } |t| > y \end{cases}$$

where $\alpha \in \mathbb{R}$ and $y > 0$. Calculate the Fourier transform $\hat{f}$ and use it to calculate the L^2 norm of the function $\dfrac{\sin t}{t}$.

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Prove that the solution $u \in C^4(\bar{\Omega})$ to the bi-harmonic equation

$$\Delta^2 u = 0 \qquad \text{in } \Omega$$
$$u = \Delta u = 0 \qquad \text{on } \partial\Omega$$

is unique.

4. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let $\gamma \in C^1(\bar{\Omega})$ be a real valued function. Given $f \in C(\partial\Omega)$, let $u \in C^2(\Omega) \cap C^1(\partial\Omega)$ be the solution to the problem

$$\operatorname{div}(\gamma(x)\nabla u(x)) = 0 \qquad \text{in } \Omega$$
$$u = f \qquad \text{on } \partial\Omega,$$

where div denotes the divergence. This defines a map $\Lambda : C(\partial\Omega) \to C(\partial\Omega)$ by

$$\Lambda(f)(x) = \gamma(x)\,\frac{\partial u}{\partial \eta}(x)$$

where $\dfrac{\partial u}{\partial \eta}$ denotes the normal derivative of u. Prove that

$$\int_{\partial\Omega} f(x)\,\Lambda(g)(x)\,d\sigma(x) = \int_{\partial\Omega} g(x)\,\Lambda(f)(x)\,d\sigma(x),$$

for all $f, g \in C(\partial\Omega)$.

HINT: if v is the solution with data g, use the divergence theorem with the fields $v\,\gamma\,\nabla u$ and $u\,\gamma\,\nabla v$, and compare the integrals obtained.

Part II. (Do 2 problems)

1. Let $f, g \in W^{1,2}(\Omega)$. Prove that $fg \in W^{1,1}(\Omega)$ and $D(fg) = fDg + gDf$.

   HINT: from Meyers-Serrin theorem there exist $f_n, g_n \in C^\infty(\Omega) \cap W^{1,2}(\Omega)$ with $f_n \to f$ and $g_n \to g$ in $W^{1,2}(\Omega)$. Show that $f_n g_n \to fg$ in $L^1(\Omega)$ and $f_n Dg_n + g_n Df_n \to fDg + gDf$ in $L^1(\Omega)$. Since f_n, g_n are smooth $D(f_n g_n) = f_n Dg_n + g_n Df_n$, conclude the result.

2. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let $\epsilon > 0$. Let $u \in C^2(\Omega \times (0, T))$ solving

$$\begin{cases} \epsilon^2 u_{tt} + u_t = \Delta_x u & \text{in } \Omega \times (0, T) \\ u(x, t) = 0 & \text{for } x \in \partial\Omega \text{ and } t > 0. \end{cases} \tag{1}$$

   Consider

$$E(t) = \int_\Omega \left(\epsilon^2 u_t^2 + |\nabla_x u|^2\right) dx.$$

   Prove that

   (a) $E(t)$ is non increasing

   (b) Let $f = f(x) \in C(\bar{\Omega})$, and $\alpha < 1$. If u satisfies (1) with $u(x, 0) = 0$ and $u_t(x, 0) = \epsilon^{-\alpha} f(x)$ for $x \in \Omega$, then deduce from (a) that

$$\int_\Omega |\nabla_x u(x, t)|^2 \, dx \to 0$$

   as $\epsilon \to 0$ uniformly for $0 \le t \le T$.

3. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a constant vector. Prove that if $u \in C^2(\bar{\Omega})$ satisfies

$$\begin{aligned} \Delta u + \alpha \cdot \nabla u - u^3 &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

   then $u \equiv 0$ in Ω.