PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

January 2002

Part I. Do three (3) of these problems.

I.1. Show that

$$\lim_{n\to\infty}\sum_{k=0}^{2n}\frac{k}{k^2+n^2}=\frac{1}{2}\ln 5.$$

I.2. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence of real numbers satisfying

(a) $\lim_{n\to\infty}\dfrac{x_1+\cdots+x_n}{n}=L$, and

(b) $\lim_{n\to\infty}\dfrac{1}{n}\sum_{k=1}^{n}k(x_k-x_{k-1})=0$; (we set $x_0=0$).

Prove that $\lim_{n\to\infty}x_n=L$.

I.3. Suppose f is Riemann integrable on $[a,b]$ and $f(x)=0$ for all $x\in[a,b]\cap\mathbb{Q}$. Prove that $\int_a^b f(x)\,dx=0$.

I.4. Let $f\in C^1[0,1]$, $\delta=\min_{[0,1]}|f'(x)|$, and $\Delta=\max_{[0,1]}|f'(x)|$. Prove that

$$\frac{1}{12}\,\delta^2\le\int_0^1 f^2(x)\,dx-\left(\int_0^1 f(x)\,dx\right)^2\le\frac{1}{12}\,\Delta^2.$$

Hint: expand $\int_0^1\int_0^1(f(x)-f(y))^2\,dx\,dy$, Fubini and mean value theorem.

Part II. Do two (2) of these problems.

II.1. Let $f_n(x) = n^{1/2}\, e^{-nx}$ on $[0,1]$. Prove that

(a) $f_n(x) \to 0$ pointwise on $(0,1]$,
(b) $\int_0^1 f_n(x)^2\, dx \le C$ for all n,
(c) f_n does not converge in $L^2(0,1)$,
(d) $\int_0^1 f_n(x)\, g(x)\, dx \to 0$ for each $g \in L^2(0,1)$.

Hint: for (d) prove it first for simple functions and then use the density of the simple functions in $L^2(0,1)$.

II.2. Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and invertible. Suppose that for each Lebesgue measurable set A, the set $g(A)$ is Lebesgue measurable and define the measure $\mu(A) = |g(A)|$. Prove that the measure μ is absolutely continuous with respect to Lebesgue measure if and only if g is an absolutely continuous function, and in that case $\dfrac{d\mu}{dx} = g'(x)$ a.e.

II.3. Let f_k be a sequence of functions in $L^2(\mathbb{R}^n)$. Suppose that $\|f_k\|_{L^2(\mathbb{R}^n)} \le M$ for all k, and $f_k \to f$ a.e. Prove that

$$\int_{\mathbb{R}^n} f_k(x)\, g(x)\, dx \to \int_{\mathbb{R}^n} f(x)\, g(x)\, dx,$$

for all $g \in L^2(\mathbb{R}^n)$.