Real Analysis Ph.D. Qualifying Exam
Temple University
August 28, 2009

Part I. (Select 3 questions.)

1. Let f be a continuous function in $[0,1]$ such that f is absolutely continuous in $[0,\epsilon]$ for every $\epsilon, 0 < \epsilon < 1$. Show that f is absolutely continuous in $[0,1]$.

2. Let $f(x) = x^2 \sin(1/x^3)$ for $x \in [-1,1]$, $x \neq 0$, and $f(0) = 0$. Show that f is differentiable on $[-1,1]$ but f' is unbounded on $[-1,1]$.

3. Let E_k be a sequence of sets. The upper limit of the sequence E_k is the set $E^* = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$. Let $\chi_E(x)$ denote the characteristic function of the set E. Prove that
$$\limsup_{k\to\infty} \chi_{E_k}(x) = \chi_{E^*}(x).$$

4. Consider the sequence $f_n(x) = n^2\, x\, e^{-n\,x^2}$ on $[1,+\infty)$. Prove that
   (a) f_n converges uniformly on $[1,+\infty)$;
   (b) f_n converges in measure on $[1,+\infty)$;
   (c) $\displaystyle\int_1^{\infty} f_n(x)\,dx \to 0$ as $n \to \infty$.

Part II. (Select 2 questions.)

1. Let $f_1, \cdots, f_k$ be continuous real valued functions on the interval $[a,b]$. Show that the set $\{f_1, \cdots, f_k\}$ is linearly dependent on $[a,b]$ over the scalar field $\mathbf{R}$ if and only if the $k \times k$ matrix with entries
$$\langle f_i, f_j \rangle = \int_a^b f_i(x)\, f_j(x)\, dx$$
has determinant zero.

2. Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of Lebesgue measurable subsets of $[0,1]$ such that $\lim_{k\to\infty} |E_k| = 1$. Prove that given $0 < \varepsilon < 1$ there exists a subsequence $\{E_{k_j}\}_{j=1}^{\infty}$ such that $\left|\bigcap_{j=1}^{\infty} E_{k_j}\right| > \varepsilon$.

   HINT: we have $\lim_{k\to\infty} |E_k^c| = 0$ and so given j there exists k_j such that $|E_{k_j}^c| < (1-\varepsilon)/2^j$. Hence $\left|\bigcup_{j=1}^{\infty} E_{k_j}^c\right| < 1 - \varepsilon$.

3. Let A be a measurable subset of $[0, 2\pi]$. Assume the Riemann-Lebesgue lemma saying that
$$\lim_{n\to\infty} \int_A \cos(nx)\, dx = \lim_{n\to\infty} \int_A \sin(nx)\, dx = 0.$$
Deduce that for each sequence x_n of real numbers we have
$$\lim_{n\to\infty} \int_A \cos^2(nx + x_n)\, dx = \frac{1}{2}|A|.$$