Mathematics Real Analysis Ph.D. Qualifying Exam
Temple University
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Part I. (Select 3 questions.)

1. Let $f \in C(\mathbf{R})$. Prove that the sequence defined by

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right)$$

converges uniformly on each finite interval $[a, b]$.

2. Let $K \subset \mathbf{R}^n$ be a compact set and let $\{O_i\}_{i \in I}$ be an open covering of K. Prove that there exists a number $\delta > 0$ (the Lebesgue number of the covering) such that for each $x \in K$ there exists O_j such that ball $B(x, \delta) \subset O_j$; $B(x, \delta)$ is the Euclidean open ball centered at x with radius δ.

   HINT: given $x \in K$ there exists $j \in I$ and $\delta_x > 0$ such that $B(x, \delta_x) \subset O_j$. Consider the following open covering of K: $\{B(x, \delta_x/2)\}_{x \in K}$. Select by compactness of K a finite sub-covering and take δ to be the minimum radius.

3. Using that $x - \sin x = \frac{1}{6}x^3 + O(x^5)$ as $x \to 0$, prove that the integral $\int_0^\infty \frac{x - \sin x}{x^{3+\alpha}}\, dx$ converges for all $0 \le \alpha < 1$.

4. Let $f_n \in C[a, b]$ with $\max_{x \in [a,b]} |f_n(x)| \le M$ for all n. Define $g_n(t) = \int_a^t f_n(x)\, dx$ for $a \le t \le b$. Prove that g_n contains a subsequence uniformly convergent in $[a, b]$.

   HINT: use Arzelá-Ascoli.

Part II. (Select 2 questions.)

1. Prove that the set of numbers in the interval $[0, 1]$ whose binary expansion has zero in all even places is a set of measure zero.

2. Let $f \in L^1(0, 1)$ and suppose that $\lim_{x \to 1^-} f(x) = A$. Prove that $(n + 1) \int_0^1 x^n f(x)\, dx \to A$ as $n \to \infty$.

3. Let $f_n \in L^2(0, 1)$ with $\|f_n\|_2 \le M$ for all n. Suppose $f_n \to f$ in measure. Prove that $f \in L^2(0, 1)$ and $\int_0^1 f_n(x)\, g(x)\, dx \to \int_0^1 f(x)\, g(x)\, dx$ for each $g \in L^2(0, 1)$.