PH.D. COMPREHENSIVE EXAMINATION
REAL ANALYSIS SECTION

August 2003

Part I. Do three (3) of these problems.

I.1. Let $f : \mathbb{R} \to \mathbb{R}$ be $C^n(\mathbb{R})$ for some $n \geq 0$. Prove that if $f^{(k)}(0) = 0$, for all $0 \leq k \leq n$, then $f^{(k)}(x) = o(|x|^{n-k})$ as $x \to 0$, for all $0 \leq k \leq n$.

I.2. Prove that on $C[0,1]$ the norms $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$ and $\|f\|_1 = \int_0^1 |f(x)|\,dx$ are not equivalent.

I.3. Prove that $\dfrac{\sin x}{x} \notin L^1(0,+\infty)$ and $\lim_{R\to\infty} \displaystyle\int_0^R \frac{\sin x}{x}\,dx$ exists.

I.4. Let $\{a_n\}_{n=1}^\infty$ be a sequence of nonnegative real numbers such that

$$a_{n+m} \leq a_n + a_m$$

for all n, m. Prove that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}.$$

Part II. Do two (2) of these problems.

II.1. Let $f_n(x) = n \sin\left(\dfrac{x}{n}\right)$. Prove that:

(a) f_n converges uniformly on any finite interval.
(b) f_n does not converge uniformly on $\mathbb{R}$.
(c) f_n does not converge in measure on $\mathbb{R}$.

II.2. Let r_n be the sequence of all rational numbers and

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{|x - r_n|^{1/2}}.$$

Prove that

(a) $\displaystyle\int_a^b f(x)\, dx < \infty$,
(b) $\displaystyle\int_a^b f(x)^2\, dx = +\infty$,

for all $a < b$.

II.3. Let $1 \leq p < \infty$, $f_n \in L^p(\mathbb{R})$ such that $f_n \to f$ a.e. Suppose that

(a) there exist $n_1 > 0$ and $A \subset \mathbb{R}$ with $|A| < \infty$ such that $\displaystyle\int_{\mathbb{R}\setminus A} |f_n(x)|^p\, dx \leq 1$ for all $n \geq n_1$; and
(b) there exist $n_0 > 0$ and $0 < \delta < 1$ such that if $|F| < \delta$, then $\displaystyle\int_F |f_n(x)|^p\, dx \leq 1$ for all $n \geq n_0$.

Then prove that $f \in L^p(\mathbb{R})$.