Ph.D. Comprehensive Examination
Real Analysis
Fall 2000

Part I. Do three of these problems.

I.1. Let $f \in C^{\infty}(\mathbb{R})$ and $f_n(x) = f^{(n)}(x)$ the derivative of order n. Suppose $f_n \to g$ uniformly on compacts. Prove that $g = C\, e^x$ for some constant C.

I.2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, nonnegative, and satisfies $f''(x) \leq 2$ in $\mathbb{R}$. Show that

(1) $|f'(x)| \leq 2\sqrt{f(x)}$ on $\mathbb{R}$.

(2) $|\sqrt{f(x)} - \sqrt{f(y)}| \leq |x - y|$ on $\mathbb{R}$.

Hint: For (1) use Taylor's theorem $f(x+t) = f(x) + t f'(x) + \frac{t^2}{2} f''(\xi)$ for all t. For (2) assume first f is strictly positive, consider $g(x) = \sqrt{f(x)}$ and use (1).

I.3. Let f, g be continuous functions in $\mathbb{R}^n$ .such that there exists $C > 0$ with

$$\{x : |f(x)| > t\} \subset \{x : |g(x)| > C\, t\}$$

for all $t > 0$. Prove that $|f(x)| \leq C^{-1} |g(x)|$.

I.4. Let $f : E \to \mathbb{R}$ be a measurable function. Prove that if $B \subset \mathbb{R}$ is a Borel set the f^{-1} is measurable.
Hint: Consider $\mathcal{A} = \{A \subset \mathbb{R} : f^{-1}(A) \text{ is measurable}\}$ and show that $\mathcal{A}$ is a σ-algebra that contains the open sets of $\mathbb{R}$.

Part II on next page

Justify your answers thoroughly. For any theorem that you wish to cite, you should either give its name or a statement of the theorem.

Part II. Do two of these problems.

II.1. Let $\alpha > 0$. Prove that

(1) if $f \in L^1(0,1)$ then

$$f_\alpha(x) = \int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

exists a.e. and is integrable on (0,1);

(2) if $f \in L^p(0,1)$ then f_α is continuous in (0,1) for $\alpha > 1/p$.

II.2. Let $1 \leq p,\ q \leq \infty,\ f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Prove tha $fg \in L^r(\mathbb{R}^n)$ with $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$.

II.3. Prove that

(1)

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(2)

$$\int_0^1 \log \frac{1}{1-x}\, dx = 1.$$