Real Analysis Exam, August 1998

Part I

Do three problems in this section

I.1. For $x \in \mathbb{R}$ let $f_n(x) = \frac{1+x^2}{1+nx^2}$. Show:

(1) f_n converges pointwise but not uniformly.
(2) $f_n \to 0$ a.e.
(3) $f_n \to 0$ in measure.

I.2. Find all values of $p \in \mathbb{R}$ for which the function $f(x) = |\log x|^p$ belongs to $L^1(0,1)$ with respect to Lebesgue measure. (Hint when $p \geq 1$: first consider integral p and use integration by parts).

I.3. Suppose $f_n : [0,1] \to \mathbb{R}$, $n = 0, 1, \ldots$ is a sequence of continuous functions such that for each $x \in [0,1]$,

(1) $f_n(x) \to 0$
(2) $f_n(x) \geq f_{n+1}(x)$

Show that $f_n \to 0$ uniformly.

I.4. Let $\Omega = \mathbb{R} \times [-1,1]$, let $f : \Omega \to \mathbb{R}$ be defined by $f(x,y) = e^{-|x|/y}/y$ for $y \neq 0$. Compute $\int_\Omega f\, dxdy$.

Part II

Do two problems in this section

II.1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y) = (2x, y/4)$. Show that if $f \in L^1(\mathbb{R}^2)$, and $f \circ T = f$ then $f = 0$ a.e.

II.2. Let $I = [0,1] \subset \mathbb{R}$. The Weierstrass approximation theorem states that if $f \in C(I)$ then there is a sequence $\{p_n\}_{n=0}^\infty$ of polynomials such that $p_n \to f$ uniformly for x in I as $n \to \infty$. Show that if $f \in C^1(I)$ then there is a sequence $\{p_n\}_{n=0}^\infty$ of polynomials converging to f in $C^1(I)$, that is, both $p_n \to f$ and $p_n' \to f'$ uniformly for x in I as $n \to \infty$.

II.3. Let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$, let $\mu = \lambda + \delta$ where λ is the Lebesgue measure and δ is the Dirac measure at 0, $\delta(E) = 1$ if $0 \in E$, $\delta(E) = 0$ if $0 \notin E$, for any $E \in \mathcal{B}$.

(1) Show that if $f \in L^\infty(\mathbb{R}, \mu)$ then $f \in L^\infty(\mathbb{R}, \lambda)$, and

$$\|f\|_{L^\infty(\mathbb{R},\lambda)} \leq \|f\|_{L^\infty(\mathbb{R},\mu)}.$$

Thus there is a well defined linear mapping $T : L^\infty(\mathbb{R}, \mu) \to L^\infty(\mathbb{R}, \lambda)$, simply taking an element f in $L^\infty(\mathbb{R}, \mu)$ and regarding it as an element of $L^\infty(\mathbb{R}, \lambda)$, and the mapping is continuous.

(2) Show that T is surjective but not injective.

Finally,

(3) Show that elements of $L^\infty(\mathbb{R}, \mu)$ have a well defined value at 0.

Hint for (2) and (3): Keep in mind that strictly speaking, the elements of $L^\infty(\mathbb{R}, \mu)$ are equivalence classes of functions.