Comprehensive Examination in Algebra
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PART I: Do three of the following problems.

1. Given a group G, recall that its *commutator subgroup* is the subgroup of G, generated by the elements $a^{-1}b^{-1}ab$, for all $a, b \in G$. Now let n be an integer ≥ 3, and let D_{2n} denote the dihedral group of order $2n$; that is, $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$. Let D'_{2n} denote the commutator subgroup of D_{2n}, and let $[D_{2n} : D'_{2n}]$ denote the index of D'_{2n} in D_{2n}

   (a) Prove that $D'_{2n} = \langle r^2 \rangle$.

   (b) Determine $[D_{2n} : D'_{2n}]$, for all n.

2. Let A be a finite abelian group with the property that for any positive integer n there exist at most n distinct elements $a \in A$ such that $a^n = 1$.

   (a) Suppose A is a p-group, where p is a prime. Prove that A is cyclic.

   (b) Prove that any finite abelian group with the above property is cyclic.

3. Let R be a principal ideal domain, and let I be an ideal of R not equal to either (0) or R itself. Prove that $I^2 \neq I$, where I^2 denotes the ideal

$$\left\{ \sum_{k=1}^{n} a_k b_k \;\middle|\; a_1, \ldots, a_n, b_1, \ldots, b_n \in I,\ n = 1, 2, \ldots \right\}.$$

4. Let F be a field and R an integral domain that contains F. Recall that an element $a \in R$ is called *algebraic over* F if there exists a polynomial $p(x) \in F[x]$ such that $p(a) = 0$. Also, recall that $F[a]$ denotes the smallest subring of R containing both F and a, that is,

$$F[a] = \{c_0 + c_1 a + ... + c_n a^n : \ c_0, ..., c_n \in F \text{ and } n \in \mathbb{N}\}$$

   Show that $F[a]$ is a field if and only if a is algebraic over F.

Part II: Do two of the following problems.

1. Let G be a finite group, H a normal subgroup of G and P a Sylow p-subgroup of G.

   (a) Show that $P \bigcap H$ is a Sylow p-subgroup of H.

   (b) Show that PH/H is a Sylow p-subgroup of G/H.

2. Let S be a not-necessarily-commutative ring with a multiplicative identity 1. Assume further that S is simple (i.e., the only two-sided ideals of S are $\langle 0 \rangle$ and S itself). Let L be a nonzero left ideal of S.

   (a) Prove that $S = LS$, where

   $$LS = \{l_1 s_1 + \cdots + l_m s_m \mid l_1, \ldots, l_m \in L,\ s_1, \ldots, s_m \in S,\ m = 1, 2, \ldots\}.$$

   (b) Prove there exist (finitely many) elements $x_1, \ldots, x_n \in S$ such that $S = Lx_1 + \cdots + Lx_n$, where

   $$Lx_1 + \cdots + Lx_n = \{l_1 x_1 + \cdots + l_n x_n \mid l_1, \ldots, l_n \in L\}.$$

3. Let F be a field and L and K finite extensions of F in an algebraic closure $\bar{F}$ of F. Let $E = LK$ be the composite of L and K.

   (a) Suppose L and K are Galois over F. Show that E is Galois over F.

   (b) Suppose further that $Gal(L/F)$ and $Gal(K/F)$ are both abelian. Show that $Gal(E/K)$ is abelian. Hint: show that two Galois transformations of E over K commute if and only if their restrictions to L commute and their restrictions to K commute.