Comprehensive Examination in Algebra
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PART I: Do three of the following problems.

1. An abelian group is *uniform* if every intersection of two nontrivial subgroups is also nontrivial.

   (a) Let A be a nontrivial finite uniform abelian group. Prove that $A \cong \mathbb{Z}/\mathbb{Z}p^n$ for some positive integer n and some prime number p.

   (b) Suppose that A is a finitely generated infinite abelian group. Prove that A is uniform if and only if A is cyclic.

   (c) Give a complete description, up to isomorphism, of the finitely generated uniform abelian groups.

2. Let $F = \mathbb{F}_3$ be the field with 3 elements and let $G = \mathrm{GL}_2(F)/F^*$ denote the group of invertible 2×2-matrices over F modulo the scalar matrices.

   (a) Show that $|G| = 24$.

   (b) Show that G acts on the set of all 1-dimensional subspaces of the vector space $V = F^2$, and only the identity element of G fixes all 1-dimensional subspaces.

   (c) Conclude that G is isomorphic to the symmetric group S_4.

3. Set $R = \mathbb{Z}[x]$, the polynomial ring in the single variable x, with integer coefficients. Prove that R is not a principal ideal domain.

4. Let F be a field and let α, β be distinct elements of F. Let $F[x]$ denote the polynomial algebra over F and define $R = \{f/g \mid f, g \in F[x], g(\alpha)g(\beta) \neq 0\}$.

   (a) Show that R is a subring of $F(x)$, the field of rational functions over F.

   (b) Determine the units of R.

   (c) Determine the maximal ideals of R.

Part II: Do two of the following problems.

1. Let G be a group of order 231.
   (a) Prove that G contains a unique Sylow 11-subgroup P.
   (b) Determine $\mathrm{Aut}(P)$, the group of automorphisms of P.
   (c) Prove that there is a group homomorphism from G into $\mathrm{Aut}(P)$.
   (d) Prove that P is contained in the center of G.

2. Let n be a positive integer, and let k be an algebraically closed field of characteristic 0. Let X and Y be $n \times n$ matrices over k such that $XY - YX = X$. Prove that X and Y have a common eigenvector. (Hint: First prove, if $v \in k^n$ is an eigenvector for X, that $v, Yv, Y^2v, \ldots$ span a vector subspace of k^n invariant under both X and Y.)

3. Let K/F be a finite Galois extension of fields, with Galois group $\Gamma = \mathrm{Gal}(K/F)$. For $\alpha \in K$, define $\mathrm{Tr}_{K/F}(\alpha) = \sum_{\gamma \in \Gamma} \gamma(\alpha)$.
   (a) Prove that $\mathrm{Tr}_{K/F}(\alpha) \in F$.
   (b) Let $x^m + cx^{m-1} + \ldots$ be the minimal polynomial of α over F. Show that m divides the degree $[K : F]$ and that $\mathrm{Tr}_{K/F}(\alpha) = -\frac{[K:F]}{m} \cdot c$.