Comprehensive Examination in Algebra
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PART I: Do three of the following problems.

1. Let G be a finite group.

   (a) Suppose that G has a normal subgroup N whose order $|N|$ and index $[G : N]$ are relatively prime. Show that N is the only subgroup of G having order $|N|$.

   (b) Suppose that $|G| = 2^k m$ (m odd) and that there is an element $g \in G$ having order 2^k. Show that G has a normal subgroup of order m. [*Hint*: Use induction on k and part (a).]

2. Let $R = \mathbb{Q}[x, y]$ be the ring of polynomials in variables x and y with rational coefficients and let
$$I = \{f(x, y) \in \mathbb{Q}[x, y] : \ f(\sqrt{2}, \sqrt{3}) = 0\}.$$

   (a) Show that I is a prime ideal of R.

   (b) Is I a maximal ideal of R? Prove or give an example of a proper ideal of R that contains I.

3. Let F be a field, and let A and B be $n \times n$ matrices with entries in F.

   (a) Show that AB and BA have the same eigenvalues.

   (b) Suppose further that B is an invertible matrix. Show that for any eigenvalue λ of AB, the λ-eigenspaces of AB and BA have the same dimensions.

4. Let R be an integral domain and $p(x)$ a monic irreducible polynomial in $R[x]$ of degree ≥ 1. Suppose $I = (p(x))$ is maximal in $R[x]$. Show that R is a field.

Part II: Do two of the following problems.

1. Let p and q be distinct primes. Show that there is no simple group of order p^2q.

2. Let R be a subring of a field F and $\alpha \in F$ an element of F. We say that α is integral over R if there exists a monic polynomial $p(x) \in R[x]$ such that $p(\alpha) = 0$. Show that α is integral over R if and only if $R[\alpha]$ is a finitely generated R-module.

3. Let n be a positive integer and F a field of characteristic relatively prime to n that contains the n^{th} roots of 1. Let $a \in F^{\times}$.

   (a) Show that $E = F(a^{1/n})$ is a Galois extension of F.

   (b) Show that $\mathrm{Gal}(E/F)$ is isomorphic to a subgroup of $\mathbb{Z}/n\mathbb{Z}$, the group of integers mod n under addition.

   (c) Let $n = p$, a prime. Show that either $E = F$ or $[E : F] = p$.