Comprehensive Examination in Algebra

Department of Mathematics, Temple University

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Part I: Do three of the following problems

1. Let $R = \mathbb{Z}[x]$, and let I be a nonzero ideal of R.

(i) Let $J = \{a \in \mathbb{Z} : a = 0$ or a is the leading coefficient of a polynomial in $I\}$. Prove that J is an ideal of $\mathbb{Z}$.

(ii) Recall, if t is a positive integer, that I^t denotes the ideal of R generated by $\{f^t : f \in I\}$. Prove that $I^t = I^{t+1}$ if and only if $I = R$.

2. Let n be a positive integer, and let X and Y be invertible $n \times n$ complex matrices such that $X^{-1}YX = e^{2\pi i/n}Y$. Determine the Jordan Form of Y.

3. Let $\mathbb{Q}^+$ denote the additive group of rational numbers, and let $\mathbb{Z}^+$ denote the additive group of integers. Prove that $\mathbb{Q}^+/\mathbb{Z}^+$ is not finitely generated.

4. Let $K = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Determine $[K : \mathbb{Q}]$.

Part II: Do two of the following problems

1. Let $G = \mathrm{GL}_2(\mathbb{F}_p)$ be the group of invertible 2×2-matrices over the field $\mathbb{F}_p$ with p elements (p a prime). Determine the number of Sylow p-subgroups of G.

2. Let R be a ring with multiplicative identity 1. Recall that a nonzero left ideal L of R is said to be *minimal* if L is simple as a left R-module.

(i) Let M be a simple left R-module, and let I be the sum of all of the minimal left ideals of R isomorphic as left R-modules to M. (In other words, I is generated by the union of all of the minimal left ideals isomorphic to M.) Prove that I is a two-sided ideal of R.

(ii) Let J denote the sum of all of the minimal left ideals of R, and assume that $J = R$. Prove that R is then the sum of some finite collection of minimal left ideals. (Hint: This conclusion does not hold true for rings without multiplicative identities.)

(iii) Prove that if C is a commutative integral domain containing a minimal left ideal then C is a field.

3. Let F/K be a finite Galois extension of fields and let $G = \mathrm{Gal}(F/K)$ be its Galois group. Furthermore, let E/K be a non-trivial subextension; so $K \subseteq E \subseteq F$ and $E \neq K$.

(i) Assume that G is nilpotent. Show that E/K contains a non-trivial Galois extension E'/K.

(ii) Assume that G is solvable and that E/K is Galois. Show that E/K contains a non-trivial Galois extension E'/K such that $\mathrm{Gal}(E'/K)$ is abelian.