PhD Algebra Exam
Spring 1989

Part I: Do three of these problems.

1. Let A be the real $3x3$ matrix all of whose entries are 1;

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find

a) the eigenvalues of A
b) for each eigenvalue, a basis for the space of eigen vectors
c) the characteristic polynomial of A
d) the minimal polynomial of A
e) the Jordan normal form of A

2. Let $\mathbb{Z}_m$ and $\mathbb{Z}_n$ be the cyclic groups of orders m and n.

a) Prove that $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $\mathrm{GCD}(m, n) = 1$.
b) Prove that every subgroup of a cyclic group is cyclic.

3. Let R be an associative ring with identity such that every element is idempotent; that is, $x^2 = x$ for all elements $x \in R$.

a) Prove that R is commutative and has characteristic 2.
b) Give two examples of such rings, one finite and one infinite.

4. True or false: Justify if true, give counterexample if false.

a) An algebraic extension of a field has finite degree.
b) A solvable group is abelian.
c) A unique factorization domain is a principal ideal domain.
d) An infinite field has characteristic zero.
e) If a group is abelian then every subgroup is normal.

Part II : Do two of these problems.

5. Let $f(x)$ be an irreducible cubic polynomial over the rationals $\mathbb{Q}$ with at least one non-real root. Let $\mathbb{K}$ be the splitting field of $f(x)$.

a) Show $[\mathbb{K} : \mathbb{Q}] = 6$
b) Show that the Galois group $G(\mathbb{K}/Q)$ is isomorphic to the symmetric group S_3 .
c) Show that there exist irreducible cubics over $\mathbb{Q}$ whose Galois groups are not isomorphic to S_3, and say what the group must be.

6. Let A be an invertible matrix over a finite field $\mathbb{F}$.

a) Show that there is an integer k such that $A^k = I$ (identity).
b) Suppose the characteristic of $\mathbb{F}$ is p, and let $a \neq 0$ be an element of $\mathbb{F}$. Find a value of k which works for the matrix

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

c) Find a value of k which works for the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}$$

7. Let p and q be primes, not necessarily distinct. Prove that any group of order p^2q is solvable; consider separately the cases $p = q$ and $p \neq q$. (You may assume Sylow theory and the class equation.)