# Comprehensive Examination in Algebra
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## Part I. Do three of these problems.

**I.1**. Let $G$ be a finite abelian group of order $n$.

(a) Suppose that $G$ has at most one subgroup of order $d$ for each $d$ dividing $n$. Show that $G$ is cyclic.

(b) Suppose that $G$ has at most $d$ elements of order $d$ for each $d$ dividing $n$. Show that $G$ is cyclic.

**I.2**. Let $G$ be a finite group and let $H \leq G$ be a subgroup of $G$. Recall that the normalizer of $H$ in $G$ is the subgroup of $G$ that is defined by $N_G(H) = \{g \in G \mid g^{-1}Hg = H\}$.

(a) Consider the action of $H$ on the set $S = \{gH \mid g \in G\}$ of left $H$-cosets by left multiplication: $h \cdot (gH) = hgH$. Show that $gH$ is a fixed point of this action (i.e., $hgH = gH$ for all $h \in H$) if and only if $g \in N_G(H)$ and use this to conclude that the index $|N_G(H) : H|$ is equal to the number of left cosets $gH$ that are fixed under left multiplication by $H$.

(b) Suppose that $H$ is a $p$-group for some prime $p$ (i.e., $|H|$ is a power of $p$). Show that

$$|N_G(H) : H| \equiv |G : H| \bmod p.$$

**I.3**. Let $R$ be a commutative ring with identity 1.

(a) Show that the principal ideal $(x)$ in the polynomial ring $R[x]$ is prime if and only if $R$ is an integral domain.

(b) Show that the principal ideal $(x^2 + 1)$ is maximal in $R[x]$ if and only if $R$ is a field and $-1$ is not a square in $R$.

**I.4**. Let $p$ be a prime and $K$ a field of characteristic other than $p$ that contains the $p^{th}$ roots of unity. Let $\alpha$ be an element in the algebraic closure of $K$ such that $\alpha^p = a \in K$. Show that $[K(\alpha) : K]$ is either 1 or $p$.

**Part II. Do two of these problems.**

**II.1**. Let $G$ be a finite group whose order is divisible by a prime $p$ but not by $p^2$. Let $\Omega$ denote the set of elements of order $p$ in $G$, and let $P \leq G$ be a Sylow $p$-subgroup of $G$.

(a) Show that

$$|\Omega| = \frac{|G|(p-1)}{|N_G(P)|} \; .$$

(b) Let $G = A_{p+1}$ be the alternating group of degree $p+1 > 3$, and let $N_G(P)$ be as in (a). Show that

$$|N_G(P)| = \frac{p(p-1)}{2} \; .$$

**II.2**. Let $V$ be a left module over some ring $R$ (not necessarily commutative) and let $f \in \mathrm{End}_R(V)$ be an endomorphism of $V$. Assume that $\mathrm{Ker}\, f^n = \mathrm{Ker}\, f^{n+1} = \mathrm{Ker}\, f^{n+2} = \dots$ and $\mathrm{Im}\, f^n = \mathrm{Im}\, f^{n+1} = \mathrm{Im}\, f^{n+2} = \dots$ holds for some positive integer $n$. Prove:

(a) $\mathrm{Im}\, f^n$ and $\mathrm{Ker}\, f^n$ are $R$-submodules of $V$, and $V = \mathrm{Im}\, f^n \oplus \mathrm{Ker}\, f^n$;

(b) the restriction of $f$ to $\mathrm{Im}\, f^n$ is an automorphism of $\mathrm{Im}\, f^n$;

(c) the restriction of $f$ to $\mathrm{Ker}\, f^n$ is a nilpotent endomorphism of $\mathrm{Ker}\, f^n$.

**II.3**. Let $K$ be the splitting field of the polynomial $x^6 - 2$ over $\mathbb{Q}$.

(a) Show that $[K : \mathbb{Q}] = 12$ and $\mathrm{Gal}(K/\mathbb{Q}) \cong D_{12}$, where $D_{12}$ is the dihedral group with 12 elements.

(b) Determine the number of subfields of $K$ of degree 6 over $\mathbb{Q}$.

(c) List explicitly (i.e., as $\mathbb{Q}$, adjoined explicit generators) all those subfields of $K$ of degree 6 over $\mathbb{Q}$ that are Galois over $\mathbb{Q}$.